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Twistor Quantisation and Curved Space-Time†

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Abstract

The formalism of twistors [the 'spinors' for the group O(2,4)] is employed to give a concise expression for the solution of the zero rest-mass field equations, for each spin ($s = 0, \frac{1}{2}, 1, \ldots$), in terms of an arbitrary complex analytic function $f(Z^\alpha)$ (homogeneous of degree $-2s-2$). The four complex variables Z^α are the components of a twistor. In terms of twistor space (C-picture) it is analytic structure which takes the place of field equations in ordinary Minkowski space-time (M-picture). By requiring that the singularities of $f(Z^\alpha)$ form a disconnected pair of regions in the upper half of twistor space, fields of positive frequency are generated.

The twistor formalism is adapted so as to be applicable in curved space-times. The effect of conformal curvature in the M-picture is studied by consideration of plane (-fronted) gravitational 'sandwich' waves. The C-picture still exists, but its complex structure 'shifts' as it is 'viewed' from different regions of the space-time. A weaker symplectic structure remains. The shifting of complex structure is naturally described in terms of Hamiltonian equations and Poisson brackets, in the twistor variables Z^α, $\bar{Z}_\alpha$. This suggests the correspondence $\bar{Z}_\alpha = \partial/\partial Z^\alpha$ as a basis for quantization. The correspondence is then shown to be, in fact, valid for the Hilbert space of functions $f(Z^\alpha)$, which give the above twistor description of zero rest-mass fields. For this purpose, the Hilbert space scalar product is described in (conformally invariant) twistor terms. The twistor expressions for the charge and the mass, momentum and angular momentum (both in 'inertial' and 'active' versions, in linearised theory) are also given.

It is suggested that twistors may supply a link between quantum theory and space-time curvature. On this view, curvature arises whenever a 'shift' occurs in the interpretation of the twistor variables Z^α, $\bar{Z}_\alpha$ as the twistor 'position' and 'momentum' operators, respectively.

1. *Introduction*

In an earlier paper (Penrose, 1967a) the formalism of *twistor algebra* was developed, which treats the geometry of Minkowski space-time from the point of view of its null line and null cone structure. In the

† This work was partly carried out during the author's five-month stay at Cornell University.

present paper this formalism is used to give a concise description of zero rest-mass fields, and the formalism is developed further so as to be applicable in curved space-times as well as flat. An unexpected interconnection between these two ideas leads to a new view of the relation between quantum theory and space-time curvature.

According to the twistor formalism, any conformally covariant operation in Minkowski space-time has a description in purely twistor terms, and with the introduction of a fixed skew-symmetric 'metric' twistor $I_{\alpha\beta}$ we can also express Poincaré covariance in purely twistor terms. The twistor algebra leads to a geometrical picture of phenomena (the C-picture) which, although strikingly different from the usual space-time description (M-picture), is nevertheless completely equivalent to it. The points (i.e. 'events') of the M-picture correspond to non-local structures ('lines'—each with the topology S^2) in the C-picture; conversely the points of the C-picture correspond to non-local structures in the M-picture (to null straight lines or, more generally, to certain twisting null line systems). An outline of the results we require here will be given in Section 2.

The motivation for rewriting physical quantities in twistor terms springs from several directions. In the first instance, there is simply the hope that when a formalism so different from the usual one is used, new insights may be gained. While it is true that certain important concepts, which were easy to express in the old formalism, can become complicated in the new (which need not be a serious drawback, since the old formalism is always at hand when required), there are other operations of great utility in the new formalism which one would be unlikely to come upon solely by considerations with the old formalism. But are these new operations likely to be of any particular importance to physics? It is here that I must be more specific and mention some of those features which motivate the specific choice of a twistor formalism for the description of space-time.

One of these features is that the twistor formalism is the natural vehicle for the *algebraic* description of *conformal invariance*. Twistors are, in fact, the 'spinors' appropriate to the six-dimensional pseudo-orthogonal group O(2,4), which is 2-1 isomorphic with the 15-parameter conformal group of (the compactified) Minkowski space-time (Cartan, 1914; Brauer & Weyl, 1935; Hepner, 1962; Murai, 1953, 1954, 1958; Segal, 1967). The connected component of the twistor group SU(2,2) is 2-1 isomorphic with connected component of O(2,4) so that twistors $\left(\text{of valence } \begin{bmatrix}1\\0\end{bmatrix}\right)$ give a four-valued four-dimensional irreducible representation of the restricted conformal group. Any other finite-dimensional representation of the conformal group can be expressed as a direct sum of twistor representations $\left(\text{of general valence } \begin{bmatrix} p \\ q \end{bmatrix}\right)$.

But is there reason to believe that the conformal group has any fundamental significance to physics?† The attitude adopted here will be that fields of zero rest-mass (which are conformally invariant) have primary significance; and that in some way, rest-mass emerges as a feature of interactions between these primary zero rest-mass fields. A simple way such an interaction might be expressed emerges in the van der Waerden description (van der Waerden, 1929) of the Dirac equation:

$$\nabla^{B'A}\phi_A = \mu\psi^{B'}, \qquad \nabla_{B'A}\psi^{B'} = -\mu\phi_A \tag{1.1}$$

Here μ is a real constant, $2^{1/2}\mu\hbar$ being the mass of the field. We may regard (1.1) as describing two neutrino-like fields ϕ_A, $\psi^{B'}$ whose free-field equations are given by putting $\mu = 0$ in (1.1). These free-field equations are then conformally invariant. We may take the view that μ is simply a coupling constant, given for all time, and that the two-field interaction (1.1) simply breaks this conformal invariance. The conformal group is then only strictly a symmetry of very high energy physics—where energies are high enough that the rest-mass interaction may be neglected (Kastrupp, 1962, 1966). Alternatively we may imagine that μ is 'really' a new variable field which for some reason (perhaps of a cosmological nature or from stability considerations, say) 'happens' to be transformable to a constant with a very high degree of accuracy. With further equations on μ, the entire set of equations can be made conformally invariant, the equations (1.1) referring to a *three*-field conformally invariant interaction. Finally, we may take the view that the interaction terms in (1.1) are of a phenomenological nature and the 'accurate' equations are really much more involved. This view is presumably what would be implied by the renormalization procedure. In addition, it is conceivable that general relativity has some significant role to play in connection with the existence of rest-mass, since gravitation is the only (known) phenomenon of nature which requires that a definite choice be made for the zero of energy, namely that it should coincide with the zero of active mass.

I do not wish to prejudice the issue here as to the ultimate nature of rest-mass. The attitude is only that it should be of significance to talk

† For a discussion of the relation of the conformal group to physics, see Fulton, T., Rohrlich, F. and Witten, L. (1962). *Rev. Mod. Phys.*, **34**, 442.

about free fields and that such fields may be viewed as having zero rest-mass. A description in terms of twistors should then have some importance as a 'background' formalism. Rest-mass could then be treated along with other interactions at a later stage.

A second feature of twistor analysis, which has been highly instrumental in the motivation for its original development, lies in the extent to which it 'geometrises' an important aspect of quantum mechanics, namely the splitting of field amplitudes into positive and negative frequency parts. In the twistor formalism, instead of resorting to Fourier analysis, it is possible to exploit an alternative description based on the positioning of singularities of analytically extended functions.† As we shall see shortly, a positive or negative frequency field will arise according as the singularities of a certain analytic (holomorphic) function representing the field, form a disconnected pair of regions in the upper or the lower half of twistor space. It is the fact that the twistors lead to a 'mild' form of 'complexification' of the space-time which enables this idea to take on a 'geometrical' significance. The geometry of the C-picture involves its *complex analytic* structure. The analytic nature of functions defined in twistor space then yields the entire structure of zero rest-mass fields in the M-picture; in particular, field equations and time-development, now become simply particular aspects of C-picture analyticity.

Some of these matters will be described in Section 3. The aim there is to show that zero rest-mass fields find a very natural and remarkably *simple* description in twistor terms. This reinforces a belief that twistors might possibly occupy a position in physics of deeper importance than just as a technical device. But if twistors really do occupy such a position, it would have to be possible to overcome one of the supreme obstacles to such a viewpoint, namely that the formalism would in principle, at least, have to be applicable in (conformally) *curved* space-times. For even if general relativity is not the correct theory of gravitation (and it is, to say the very least, the best theory of gravitation available at the present time), there can be little doubt that the conformal structure of space-time, as defined by its null cones, *does differ* from that of Minkowski space-time. (A critical feature of theories of gravitation based on a conformally flat null-cone structure, e.g. Nordström's theory, is that there is *no* resultant bending of light by a massive body. It may be taken that this, at least, is experimentally disproved.)

The essential difficulty involved in attempting to adapt the twistor

† See, for example, Streater, R. F. and Wightman, A. S. (1964). *PCT, Spin and Statistics and All That.* W. A. Benjamin, New York.

formalism to conformally curved space-time lies in the fact that the *complex analytic structure* of the C-picture is *destroyed by conformal curvature* of the M-picture (Section 4). This would seem, at first sight, to invalidate completely the use of twistors for conformally curved manifolds, since it is precisely the complex analytic structure of the C-picture which gives rise to all its important properties. However, it is here that twistors supply an unexpected *link between quantum mechanics and space-time curvature*. A non-analytic transformation of twistor space is one which mixes up the twistor coordinates Z^α with their complex conjugates $\bar{Z}_\alpha$. But the precise type of non-analytic transformation of the C-picture which is induced by the presence of conformal curvature in the M-picture turns out to be one preserving Poisson brackets, where $\bar{Z}_\alpha$ is regarded as the canonical conjugate of Z^α. This suggests that in the passage to quantum theory, Z^α should be regarded as an operator, where the operator $\bar{Z}_\alpha$ is identified with $\partial/\partial Z^\alpha$. (Planck's constant will be absorbed into the definition of the twistor variables Z^α.) With this identification, one can still operate with analytic functions in twistor space. What in the 'classical' theory was a 'canonical' non-analytic transformation of twistor space, corresponds in the quantized (one-particle) theory, to linear transformation of the *space of analytic functions* defined on twistor space. In fact, with the appropriate choice of norm, these become unitary transformations of a Hilbert space. The operations Z^α and $\partial/\partial Z^\alpha$ now become explicit operations in the M-picture which apply to zero rest-mass fields. The operator Z^α lowers the spin by $\frac{1}{2}\hbar$, while $\partial/\partial Z^\alpha$ raises it by $\frac{1}{2}\hbar$. The identification of $\bar{Z}_\alpha$ with $\partial/\partial Z^\alpha$ becomes explicitly consistent with the Hilbert space scalar product. Thus, we must regard the analytic structure of the C-picture as being relevant to the *quantum* structure of fields (e.g. to one-particle states), while the non-analytic transformations induced by curvature in the M-picture refer essentially to the *classical limit*.

We are led to a new view of the nature of space-time curvature. Let us adopt the attitude that in a certain region of space-time, the twistor variables Z^α are the ones, in terms of which physical quantities are to be expressed. The Z^α are the C-space 'position' operators and the $\bar{Z}_\alpha = \partial/\partial Z^\alpha$ are the conjugate 'momenta'. As we pass through a region of conformal curvature (e.g. a gravitational wave) the 'position' and 'momentum' variables get mixed up. We may take a 'self-consistent' or 'Machian' view of the correspondence principle, whereby the particular choice of quantum variables which are to be regarded as 'position' variables is governed by the nature of the large-scale physical structures in the region under consideration. Then, as we move to another region of space-time, the influence of a change in the large-scale structures is to effect a 'shift' in the natural interpretation of the variables. Thus, curvature of space-time emerges as a phenomenon intimately connected with such an attitude to the correspondence principle and with the quantum structure of nature.

2. *Outline of Twistor Algebra*

The main features of twistor algebra that will be required here will now be reviewed. For further details, the reader is referred to an earlier paper (Penrose, 1967a). Let x^0, x^1, x^2, x^3 be standard Minkowski coordinates in flat space-time M—metric given by

$$ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$$

Introduce a 2-spinor notation,† relating $x^{AA'}$, $x_{AA'}$ to x^a by

$$x^{00'} = x_{11'} = 2^{-1/2}(x^0 + x^1), \qquad x^{01'} = -x_{10'} = 2^{-1/2}(x^2 + ix^3) \quad (2.1)$$
$$x^{10'} = -x_{01'} = 2^{-1/2}(x^2 - ix^3), \qquad x^{11'} = x_{00'} = 2^{-1/2}(x^0 - x^1)$$

When the coordinates x^a are real, the matrices $(x^{AA'})$ and $(x_{AA'})$ are Hermitian so we have two real coordinates u, v and one complex coordinate ζ, given by

$$u = x^{00'}, \qquad v = x^{11'}, \qquad \zeta = x^{01'}, \qquad \bar{\zeta} = x^{10'} \quad (2.2)$$

The metric of M now takes the form

$$ds^2 = 2\,du\,dv - 2\,d\zeta\,d\bar{\zeta} \quad (2.3)$$

A twistor Z^α, of valence $\begin{bmatrix}1\\0\end{bmatrix}$ is a quantity with four complex components Z^0, Z^1, Z^2, Z^3, its complex conjugate $\bar{Z}_\alpha$ being a twistor of valence $\begin{bmatrix}0\\1\end{bmatrix}$, related to Z^α by

$$\bar{Z}_0 = \overline{Z^2}, \qquad \bar{Z}_1 = \overline{Z^3}, \qquad \bar{Z}_2 = \overline{Z^0}, \qquad \bar{Z}_3 = \overline{Z^1} \quad (2.4)$$

(When the bar extends only over the kernel symbol, this refers to the twistor complex conjugation operation. When the bar extends over both the kernel symbol and its index, then this means simply the complex conjugate of the complex number involved.) The twistor Z^α

† The primed index letter A' must be regarded as a distinct letter from A, so that no contraction is implied in $X^{AA'}$. Under complex conjugation, unprimed indices become primed and primed indices become unprimed.

is called *right-handed*, *left-handed*, or *null* respectively, according as the scalar

$$Z^\alpha \bar{Z}_\alpha = Z^0\overline{Z^2} + Z^1\overline{Z^3} + Z^2\overline{Z^0} + Z^3\overline{Z^1} \tag{2.5}$$

is positive, negative, or zero. (The summation convention will be employed throughout.) A null twistor Z^α describes a *null straight line* Z in M according to the scheme:

$$Z^0{:}Z^1{:}Z^2{:}Z^3 = du{:}d\bar{\zeta}{:}i\zeta\, d\bar{\zeta} - iv\,du{:}i\bar{\zeta}\,du - iu\,d\bar{\zeta} \tag{2.6}$$

where u, v, ζ are coordinates [cf. (2.2), (2.1)] of some point P on Z and $du{:}dv{:}d\zeta$ define the direction of Z. Since Z is a null line we have, by (2.3),

$$du\,dv = d\zeta\, d\bar{\zeta} \tag{2.7}$$

Hence Z^α, being invariant under $x^a \rightarrow x^a + k\,dx^a$, is independent of the choice of P on Z. By (2.7), we can also write

$$Z^0{:}Z^1{:}Z^2{:}Z^3 = d\zeta{:}dv{:}i\zeta\, dv - iv\,d\zeta{:}i\bar{\zeta}\,d\zeta - iu\,dv \tag{2.8}$$

Note that it is Z^α *up to a complex factor of proportionality* which uniquely corresponds to the line Z. When $Z^0 = Z^1 = 0$, we do not get a finite line in M, but rather a generator of the *null cone at infinity* for M. If we admit these lines *at infinity* as part of the conformal structure of M ('compactified' Minkowski space-time), then a null twistor Z^α (up to proportionality) only fails to define a unique null line in M if $Z^\alpha = 0$.

If X^α and Y^α are null twistors, defining null lines X and Y in M, respectively, then the condition for X and Y to meet (possibly at infinity) is

$$X^\alpha \bar{Y}_\alpha = 0 \tag{2.9}$$

(This is, of course, the same as $Y^\alpha \bar{X}_\alpha = 0$.) If X and Y do meet, at P, say, then any twistor Z^α which represents a generator of the *null cone* with vertex P has the form

$$Z^\alpha = \lambda X^\alpha + \mu Y^\alpha \tag{2.10}$$

This is necessarily null, by (2.9). If X and Y 'meet at infinity' (but do not both lie at infinity) then the null cone becomes a null hyperplane, with vertex P on the null cone at infinity. If both X and Y lie at infinity, then P becomes the vertex, I, of the null cone at infinity itself. Thus any point P in M, including those which lie at infinity, can be represented in twistor terms by a linear set of the type of (2.10). In fact we can use the twistor $P^{\alpha\beta} = X^\alpha Y^\beta - Y^\alpha X^\beta$, of valence $\begin{bmatrix}2\\0\end{bmatrix}$, to represent such a linear set, if desired. We can even represent any point of *complexified* Minkowski space-time M^* by a linear set like (2.10), but where not all the Z^α are null.†

We have, available, two alternative geometrical pictures for the description of phenomena (Fig. 1), namely the M-picture (which is the normal space-time description) and the C-picture, the space C

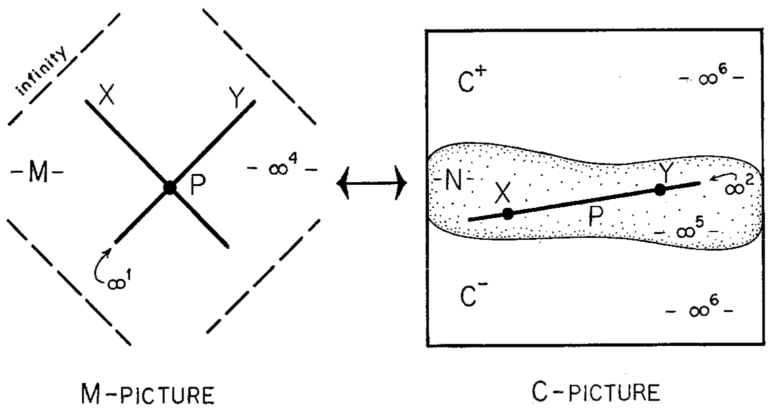

Figure 1.—The M-picture and C-picture representations of the twistors X^α, Y^α and $P^{\alpha\beta} = X^\alpha Y^\beta - Y^\alpha X^\beta$, where $X^\alpha \bar{X}_\alpha = Y^\alpha \bar{Y}_\alpha = 0 = X^\alpha \bar{Y}_\alpha$.

being the three-complex-dimensional projective space of proportionally classes of twistors Z^α, of valence $\begin{bmatrix}1\\0\end{bmatrix}$. As a *real* manifold, the space C is six-dimensional. A five-real-dimensional submanifold N (topology $S^2 \times S^3$) of C defines the null twistors Z^α ($Z^\alpha \bar{Z}_\alpha = 0$). The removal of N from C leaves two disconnected open subsets C^+ and C^- (each with topology $S^2 \times E^4$) of C, defining, respectively, the right-handed ($Z^\alpha \bar{Z}_\alpha > 0$) and left-handed ($Z^\alpha \bar{Z}_\alpha < 0$) twistors Z^α. The manifold N may be thought of as the space of null lines in M (compactified, so M has topology $S^1 \times S^3$). We may refer to C^+ and C^- as spaces of 'complexified' null lines in M, but we must bear in mind that this 'complexification' process only increases the (real) dimensionality of the null line system from five to six. The points of M are represented in the C-picture as *complex projective lines* (topology S^2), which lie entirely within N. A projective line P, in C, which does not

† In fact, M^* is just the *Grassmannian* (or Klein representation) of projective lines in the complex projective three-space C. (See, for example, Todd, J. A. (1947). *Projective and Analytical Geometry*. London; Semple, J. G. and Roth, L. (1949). *Introduction to Algebraic Geometry*. Oxford.

lie entirely on N represents a point of the *complexified* space M^*, for which the coordinates x^a are not all real. In fact, the *imaginary part* of x^a is spacelike, null, or timelike, respectively, according as P intersects N in a one-real-dimensional region (a curve: topology S^1), in a point, or not at all. If null or timelike, the imaginary part of x^a is future-pointing or past-pointing according as P lies in $C^+ \cup N$ or $C^- \cup N$.

A twistor W_α of valence $\begin{bmatrix}0\\1\end{bmatrix}$ defines a complex projective plane W in C (a four-real-dimensional submanifold of C), namely the set of points Z, of C, satisfying

$$W_\alpha Z^\alpha = 0 \tag{2.11}$$

Conversely, the plane W defines W_α uniquely up to proportionality. Now the plane W meets N in a three-real-dimensional set of points in the C-picture. These represent, in the M-picture a three-dimensional system of null lines (a null congruence). Such a congruence will define W uniquely. When W_α is null ($W_\alpha \overline{W}^\alpha = 0$), the W-congruence is simply the system of null lines in M meeting the null line $\overline{W}$ (with coordinates $\overline{W}^\alpha$). When W_α is right-handed ($W_\alpha \overline{W}^\alpha > 0$) [*resp*. left-handed ($W_\alpha \overline{W}^\alpha < 0$)], the W-congruence is a system of null lines in M, one through each point of M, which *twists* about every point in a *right-handed* [*resp. left-handed*] sense (a Robinson congruence). Thus, by invoking twistor complex conjugation, we can represent any twistor Z^α of valence $\begin{bmatrix}1\\0\end{bmatrix}$, up to proportionality, whether Z^α is null or not, by a (generally twisting) congruence of null lines in M, namely that defined by $W_\alpha = \bar{Z}_\alpha$.

We may think of twistor complex conjugation as defining a duality correspondence in the projective space C (a Hermitian correlation). To any point Z in C corresponds a unique 'polar' plane $\bar{Z}$; to any plane W in C corresponds a unique point $\overline{W}$ (the 'pole' of W). The point Z lies on the plane W if and only if $\overline{W}$ lies on $\bar{Z}$. The set N consists precisely of those points Z which lie on their 'polar' planes $\bar{Z}$. Dually, the plane W 'touches' N if and only if it contains its 'pole' $\overline{W}$. Similarly, a projective line P in C has a uniquely defined 'polar' line $\overline{P}$; Z lies on P if and only if $\overline{P}$ lies on $\bar{Z}$. The correspondence $P \leftrightarrow \overline{P}$ between lines in the C-picture represents, in the M-picture, precisely the correspondence between a point and its complex conjugate in the complexified space M^*. The *real* points of M^*, namely the points of M, are represented in the C-picture by the lines lying entirely on N. These are just the lines for which $P = \overline{P}$. More generally, the complex conjugation operation will apply to *any* twistor $K^{\alpha\beta\ldots\phi}_{\rho\ldots\tau}$ of general valence $\begin{bmatrix} p \\ q \end{bmatrix}$. The result is a twistor $\bar{K}^{\rho\ldots\tau}_{\alpha\beta\ldots\phi}$ of valence $\begin{bmatrix} q \\ p \end{bmatrix}$, where the labellings of the 0 and 1 components are interchanged with those of the 2 and 3 components on complex conjugation in the manner of (2.4); e.g. $\bar{K}^{2\ldots1}_{03\ldots2} = \overline{K^{21\ldots0}_{0\ldots3}}$ etc. In each case, any geometrical interpretation for $K^{\alpha\beta\ldots\phi}_{\rho\ldots\tau}$ in the C-picture will give rise to a corresponding *dual* interpretation for $\bar{K}^{\rho\ldots\tau}_{\alpha\beta\ldots\phi}$.

The allowable twistor transformations, other than those which involve a complex conjugate operation or a space or a time reversal in the M-picture, are given by

$$\tilde{K}^{\alpha\beta\ldots\phi}_{\rho\ldots\tau} = K^{\psi\zeta\ldots\theta}_{\lambda\ldots\nu}\, t_\psi{}^\alpha\, t_\zeta{}^\beta \ldots t_\theta{}^\phi\, \bar{t}_\rho{}^\lambda \ldots \bar{t}_\tau{}^\nu \tag{2.12}$$

where

$$t_\beta{}^\alpha \bar{t}_\gamma{}^\beta = \delta_\gamma{}^\alpha = \bar{t}_\beta{}^\alpha t_\gamma{}^\beta; \qquad |t_\beta{}^\alpha| = 1 \tag{2.13}$$

Because of the twistor complex-conjugation rule (here applied to $t_\beta{}^\alpha$), it follows that the group of $t_\beta{}^\alpha$-matrices satisfying (2.13) is just SU(2, 2). (The Hermitian form (2.5) has signature $(+, +, -, -)$.) In the C-picture, the transformations (2.12) (regarded as active transformations) are simply the projective transformations of C which leave N invariant and do not interchange C^+ with C^-. In the M-picture these are the conformal transformations of M continuous with the identity.

Twistors can also be represented (a little more completely) in the M-picture in terms of certain spinor fields. If T_α is any twistor of valence $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we can define a 2-spinor field $\tau_A(x^a)$ by

$$\tau_A = \kappa_A - ix_{AB'}\rho^{B'} \tag{2.14}$$

where $T_0 = \kappa_0$, $T_1 = \kappa_1$, $T_2 = \rho^{0'}$, $T_3 = \rho^{1'}$. Then τ_A transforms correctly as a conformal density of weight $\frac{1}{2}$ as T_α undergoes (2.12). In fact, a spinor field of the type given by (2.14) (where κ_A and $\rho^{B'}$ are constant) is the general solution of the equation

$$\nabla_{P'(A}\tau_{B)} = 0 \tag{2.15}$$

where round brackets denote symmetrisation and where

$$\nabla_{P'A} \equiv \partial/\partial x^{AP'} \tag{2.16}$$

(so, equivalently, $\nabla^{P'A} \equiv \partial/\partial x_{AP'}$). The field τ_A gives a conformally invariant M-picture realisation of the twistor T_α up to a multiple of ±1, $\pm i$. (This fourfold ambiguity arises because the spinor field τ_A picks up a factor $\pm i$ as it crosses the null cone at infinity. Such an ambiguity is essential because of the four-valued nature of the $\begin{bmatrix}0\\1\end{bmatrix}$-twistor representation of the conformal group.) Any completely symmetric covariant twistor $S_{\alpha\beta\ldots\phi} = S_{(\alpha\beta\ldots\phi)}$ also has an M-picture realization as a spinor field somewhat similar to (2.14). For example, for valence $\begin{bmatrix}0\\2\end{bmatrix}$, if (for $A, B = 0, 1$) we put $S_{AB} = \lambda_{AB}$, $S_{A,B+2} = \mu_A^{B'}$, $S_{A+2,B+2} = \nu^{A'B'}$ (where $S_{\alpha\beta} = S_{\beta\alpha}$ so $\lambda_{AB} = \lambda_{BA}$, $\nu^{A'B'} = \nu^{B'A'}$), the spinor field

$$\sigma_{AB} = \lambda_{BA} - 2i\mu_{(A}^{C'} x_{B)C'} - \nu^{C'D'} x_{AC'} x_{BD'} \tag{2.17}$$

is of the type of the general symmetric solution of

$$\nabla_{P'(A}\sigma_{BC)} = 0 \tag{2.18}$$

and represents $S_{\alpha\beta}$ in a conformally invariant way up to sign ± 1. A general (non-symmetric) twistor of valence $\begin{bmatrix}p\\q\end{bmatrix}$ has a representation similar to (2.17), but only as a many-point field.

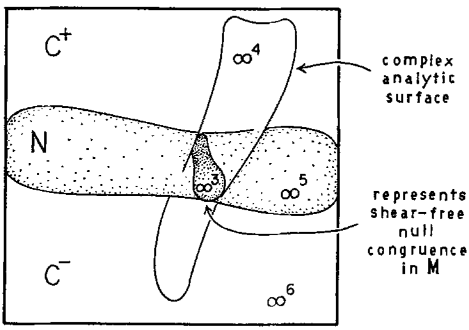

Figure 2.—The Kerr theorem.

The geometrical significance to the M-picture of the analytic structure of the C-picture is best illustrated by the *theorem of Kerr* (Kerr, unpublished). Let Q be a complex analytic surface in C, so Q is defined by the vanishing of a homogeneous analytic function in the twistor coordinates Z^α. The set Q is four-real-dimensional and intersects N in a three-real-dimensional region (Fig. 2). This defines a congruence of null lines in M. Kerr's theorem states that such a congruence is necessarily *shear-free* and conversely that any shear-free congruence of null lines in M is obtainable in this way (or as a limiting case of such a construction). The shear-free condition states that if a 'bundle' of null lines of the congruence neighbouring a given null line X, and lying (to first order) in the null hyperplane through X, has a (small) circular cross-section at one point of X, then it also has a (small) circular cross-section at every other point of X (Fig. 3). (The Robinson congruences, given when Q is a plane, are particular examples with this property.) The shear-free condition appears as a kind of M-picture realisation of the Cauchy-Riemann equations for C.

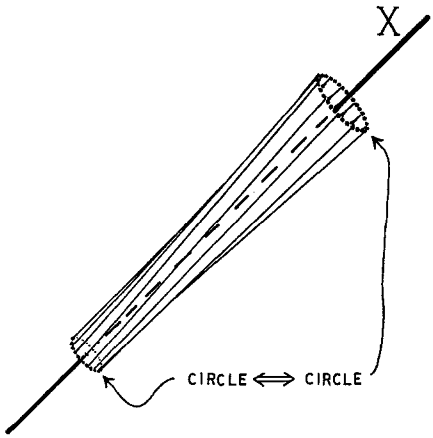

Figure 3.—The shear-free condition. (The circles lie, to first order, in the null hyperplane containing X.)

3. *Twistor Description of Zero Rest-Mass Fields*

In another paper (Penrose, 1968), the following contour integral formula was introduced, which expresses the general (analytic) solution of the spin s, zero rest-mass, free-field equations, for

$$s = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots$$

in terms of an arbitrary analytic function of three complex variables $F(\lambda, \mu, \nu)$:

$$\phi_r = \frac{1}{2\pi i} \oint \lambda^r F(\lambda, u + \lambda\bar{\zeta}, \zeta + \lambda v)\, d\lambda \quad (r = 0, 1, \ldots, 2s) \qquad (3.1)$$

(The contour is to surround singularities of F and to vary continuously with u, v, ζ.) For, we have $\partial\phi_r/\partial\bar{\zeta} = \partial\phi_{r+1}/\partial u$ and $\partial\phi_r/\partial v = \partial\phi_{r+1}/\partial\zeta$ $(r = 0, \ldots, 2s - 1)$, if $s > 0$, and $\{\partial^2/\partial u\,\partial v - \partial^2/\partial\zeta\,\partial\bar{\zeta}\}\phi_0 = 0$ for $s = 0$. Writing

$$\phi_0 = \phi_{00\ldots0}, \qquad \phi_1 = \phi_{10\ldots0}, \qquad \ldots, \qquad \phi_{2s} = \phi_{11\ldots1} \qquad (3.2)$$

these equations become, if $s > 0$,

$$\nabla^{P'A}\phi_{AB\ldots L} = 0 \tag{3.3}$$

where $\phi_{AB\ldots L}$ is symmetric ($= \phi_{(AB\ldots L)}$) with $2s$ indices and

$$\nabla_{P'A}\nabla^{P'A}\phi = 0 \tag{3.4}$$

if $s = 0$. Equations (3.3) and (3.4) are simply the zero rest-mass free-field equations (Dirac, 1936; Fierz, 1940) for spin $s > 0$ (Dirac-Fierz) and spin $s = 0$ (D'Alembert), respectively.

Thus the free-field equations on $\phi_{AB\ldots L}$ are an automatic consequence of the analyticity of F. The converse result that *any* analytic zero rest-mass free field $\phi_{AB\ldots L}$ has a representation in the form (3.1) is also true, but the argument giving the construction of F from $\phi_{A\ldots L}$ will not be entered into here. We note, in this context, that the function F is not, however, uniquely determined by $\phi_{A\ldots L}$. If, for example, we add to F any function which is regular inside the contour (for every u, v, ζ), then clearly the resulting field $\phi_{A\ldots L}$ will be unaffected. This freedom of choice for F defines a kind of gauge group (different from the usual gauge group) but which depends on the positioning of the contour (as a function of u, v, ζ). The nature of this gauge freedom will not be discussed here. Nor will certain interesting features of the representation (3.1), such as the fact that 'null' or 'algebraically special' solutions of (3.3) can be very readily generated, by merely requiring that the contour surround only a simple or low-order pole of F (Penrose, 1968). Instead, two matters of more immediate relevance will be treated, namely the transcription of (3.1) into a more general twistor form and a topological requirement on the singularities of F that ensures that the field $\phi_{A\ldots L}$ has positive (or, alternatively negative) frequency. The question of a conformally invariant Hilbert space norm for fields $\phi_{A\ldots L}$, given in terms of F, will be discussed in Section 5.

We note, first, that using the notation (2.6) or (2.8) we can write

$$F(\lambda, u + \lambda\bar{\zeta}, \zeta + \lambda v) \equiv F\left(-\frac{Z^0}{Z^1}, \frac{iZ^3}{Z^1}, -\frac{iZ^2}{Z^1}\right) \tag{3.5}$$

where

$$du + \lambda\, d\bar{\zeta} = 0 = d\zeta + \lambda\, dv \tag{3.6}$$

Set

$$f(Z^\alpha) \equiv f(Z^0, Z^1, Z^2, Z^3) \equiv (iZ^1)^{-2s-2} F\left(-\frac{Z^0}{Z^1}, \frac{iZ^3}{Z^1}, -\frac{iZ^2}{Z^1}\right) \tag{3.7}$$

Then $F(Z^\alpha)$ is analytic and *homogeneous* of degree $-2s-2$ in Z^α. The twistor Z^α represents a null line Z through the point P with coordinates u, v, ζ. The direction of Z is defined by λ according to (3.6). Let X and Y be the two particular null lines through P given respectively by $\lambda = \infty$ and $\lambda = 0$. We can assign the twistors X^α, Y^α to X, Y as follows:

$$(X^\alpha) = (i, 0, v, -\bar{\zeta}); \qquad (Y^\alpha) = (0, -i, \zeta, -u) \tag{3.8}$$

and set

$$Z^\alpha = \lambda X^\alpha + Y^\alpha \tag{3.9}$$

This is consistent with (2.6) and (3.6), and we have $Z^1 = -i$. By (3.7) and (3.5), we can now write the formula (3.1) as

$$\phi_r(X^\alpha, Y^\alpha) = \frac{1}{2\pi i}\oint \lambda^r f(\lambda X^\alpha + Y^\alpha)\, d\lambda \quad (r = 0, 1, \ldots, 2s) \tag{3.10}$$

The quantities ϕ_r are here written explicitly as functions of X^α and Y^α. This is because the spinor components (3.2) are dependent on a particular choice of spin frame, and although this spin frame was originally related to the coordinate system (2.1), we can think of X^α and Y^α, instead, as defining the spin frame at P. Indeed, we can now dispense with our original coordinates (2.1) altogether, since the point P is defined by X^α and Y^α via

$$P^{\alpha\beta} = X^\alpha Y^\beta - Y^\alpha X^\beta \tag{3.11}$$

which serve as the twistor coordinates for P. We can, in fact, enlarge the domain of the formula (3.10) by dropping the requirement that X^α and Y^α have the special form (3.8), retaining merely (for a real point P) the condition that X^α and Y^α both be null and satisfy $X^\alpha \bar{Y}_\alpha = 0$. Keeping the $P^{\alpha\beta}$ of (3.11) fixed, the freedom of choice for X^α and Y^α is given by the 'spin transformation':

$$\hat{X}^\alpha = \rho X^\alpha + \sigma Y^\alpha, \qquad \hat{Y}^\alpha = \tau X^\alpha + \omega Y^\alpha \tag{3.12}$$

with

$$\rho\omega - \sigma\tau = 1 \tag{3.13}$$

where inversely

$$X^\alpha = \omega \hat{X}^\alpha - \sigma \hat{Y}^\alpha, \qquad Y^\alpha = -\tau \hat{X}^\alpha + \rho \hat{Y}^\alpha \tag{3.14}$$

Substituting (3.14) into (3.10), putting

$$\lambda = (\rho\mu + \tau)/(\sigma\mu + \omega) \tag{3.15}$$

and keeping the contour 'fixed' (i.e. so that the same values of $f(Z^\alpha)$ are involved after the substitution), we get

$$\phi_r(X^\alpha, Y^\alpha) = \frac{1}{2\pi i}\oint (\rho\mu + \tau)^r(\sigma\mu + \omega)^{2s-r} f(\mu \hat{X}^\alpha + \hat{Y}^\alpha)\, d\mu \tag{3.16}$$

When expressed in terms of

$$\phi_r(\hat{X}^\alpha, \hat{Y}^\alpha) = \frac{1}{2\pi i}\oint \mu^r f(\mu\hat{X}^\alpha + \hat{Y}^\alpha)\,d\mu \quad (r = 0, 1, \ldots, 2s) \tag{3.17}$$

(3.16) yields the transformation law for a $D(s,0)$ representation† of the homogeneous Lorentz group at P. This is precisely what is required of the formula (3.10), in order that it should correspond to a field $\phi_{A\ldots L}$ correctly transforming under local Lorentz transformations. It is for this purpose that the function f, as defined in (3.7), is chosen to be *homogeneous of degree* $-2s-2$. With the original expression (3.1), the role played by the value of s does not emerge. The full twistor expression (3.10), on the other hand, contains the *entire transformation behaviour* of the field.

We may note the general twistor form of the equations (3.3):

$$\frac{\partial\phi_r}{\partial X^\alpha} = \frac{\partial\phi_{r+1}}{\partial Y^\alpha} \quad (r = 0, 1, \ldots, 2s-1) \tag{3.18}$$

and, for $s = 0$, of (3.4):

$$\frac{\partial^2\phi_0}{\partial X^\alpha\,\partial Y^\beta} = \frac{\partial^2\phi_0}{\partial Y^\alpha\,\partial X^\beta} \tag{3.19}$$

(The twistor 'wave equation' (3.19) applies also, of course, if $s > 0$ with ϕ_0 replaced by any of the ϕ_r.) Each ϕ_r is separately homogeneous of degree $-(r+1)$ in X^α and degree $-(2s-r+1)$ in Y^α, so that by Euler's theorem

$$X^\alpha\frac{\partial\phi_r}{\partial X^\alpha} = -(r+1)\,\phi_r, \qquad Y^\alpha\frac{\partial\phi_r}{\partial Y^\alpha} = -(2s-r+1)\,\phi_r \tag{3.20}$$

Thus (3.18) or (3.19), together with (3.20), are the *twistor versions of the zero rest-mass free-field equations*. Note that in twistor space, a field (with spin) is regarded as a *two-point function*. The *line* P in the C-picture which connects the *two points* X and Y (of C) corresponds, in the M-picture, to the *point* P at which the field $\phi_{A\ldots L}$ is evaluated, this being the intersection of the two *null lines* X and Y. The dependence of the ϕ_r on the positioning of the points X, Y on P (C-picture) corresponds to the *spin structure* of the field (this being given, in the M-picture, by the dependence of $\phi_{A\ldots L}$ on choice of spin frame at the point P—as defined by X^α, Y^α). Note that if $s = 0$, then ϕ is actually independent of X^α and Y^α except in so far as they define $P^{\alpha\beta}$

† See, for example, Ya Liubarski, G. (1960). *The Application of Group Theory in Physics* (trans. Dedijer, S.). Pergamon Press, New York; or Streater, R. F. and Wightman, A. S. (1964). *PCT, Spin and Statistics and All That*. W. A. Benjamin, New York.

according to (3.11). (This follows† from $X^\alpha \partial\phi_0/\partial Y^\alpha = 0 = Y^\alpha \partial\phi_0/\partial X^\alpha$, which is a consequence of (3.19) and (3.20).) This simply reflects the scalar nature of the field in this case.

The formula (3.10) can also be applied when X^α and Y^α are not null and need not satisfy $X^\alpha \bar{Y}_\alpha = 0$. Then the line P of the C-picture, which connects the two points X and Y [$P^{\alpha\beta}$ given as in (3.11)], does not lie on N and so represents a *complex* point $P \in M^*$ in the M-picture. The M-picture realisation of (3.10) in its full generality is thus a *complexified* zero rest-mass field. It is this fact that enables us to give a direct C-picture interpretation of a condition which ensures that $\phi_{A\ldots L}$ is a positive (or alternatively negative) frequency field. (We recall that *any* C-picture property has an interpretation in terms of the *real* space M. Thus, even the complex points of M^* can be interpreted in terms of real structures (Penrose, 1967a) in M. It is interesting to note, in this context, that much of the structure of $f(Z^\alpha)$ has a very *direct* 'real' interpretation in the M-picture. For *null* Z^α we may think of $f(Z^\alpha)$ as defining a function on the space of null lines of M, although the dependence of f on the scaling of Z^α must be borne in mind. This function enters directly into defining the field $\phi_{A\ldots L}$ at real points P and also at some complex points P.)

Let us consider the nature of the condition of 'positive frequency' as applied to a field $\phi_{A\ldots L}$. I shall be concerned only with fields which are non-singular throughout the entire (conpactified) space-time M. This will imply, in particular, that the field enjoys suitable asymptotic behaviour (i.e. in the neighbourhood of the null cone at infinity), yielding the 'peeling-off' property for the field (Penrose, 1965). In this context we must take into account the phenomenon of Grgin (1966): for a free-wave field we must expect the field to jump by a factor $(\pm i)^{2s+2}$ as we cross the null cone at infinity. Owing to the four-valued nature of the M-picture realisation of twistors, this is indeed the type of behaviour that we should expect from (3.10). I shall, in fact, regard the field $\phi_{A\ldots L}$ as being appropriately 'analytic across infinity' *only* if it possesses the correct Grgin discontinuity there. Then the fields under consideration will be imploding-exploding free-waves which tail off suitably towards infinity in all directions. In twistor terms, this condition is simply that $\phi_r(X^\alpha, Y^\alpha)$ should be regular for all null X^α, Y^α for which $X^\alpha \bar{Y}_\alpha = 0$.

We can imagine the field $\phi_{A\ldots L}$ decomposed into plane waves. The various amplitudes of these waves will be given by the *null-datum* (Penrose, 1963, 1967b) for $\phi_{A\ldots L}$ at all the various points on the null

† See, for example, Hodge, W. D. and Pedoe, D. (1952). *Methods of Algebraic Geometry*, Vol. II, Cambridge University Press.

cone at infinity. The null-datum for $\phi_{A\ldots L}$ at a point P, for a null hypersurface containing a null line X through P, is simply

$$\phi \equiv \phi(X^\alpha; P^{\alpha\beta}) \equiv \phi_0(X^\alpha, Y^\alpha) \tag{3.21}$$

where $P^{\alpha\beta}$ is related to X^α, Y^α by (3.11). (The fact that $\phi_0(X^\alpha, Y^\alpha)$ depends on X^α and $P^{\alpha\beta}$, and not on Y^α, is a consequence of (3.14) and (3.16), with $\sigma = 0$, $\omega = 1$.) The amplitude for the plane-wave component of $\phi_{A\ldots L}$ corresponding to a *fixed* null direction is given by (3.21) where X^α is kept fixed and $P^{\alpha\beta}$ is allowed to vary. Here X is

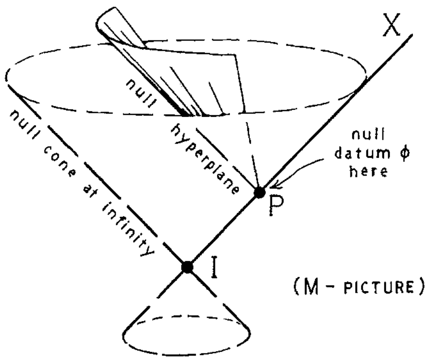

Figure 4.—A null hyperplane of constant amplitude, in a plane wave, has a vertex P on the null cone at infinity. The amplitude is measured by the complex null datum $\phi(X^\alpha; P^{\alpha\beta})$.

the generator of the null cone at infinity which corresponds to this fixed null direction (Fig. 4); P is a variable point on X which defines a *null hyperplane of constant amplitude* for the wave. (P is the 'vertex' of this null hyperplane.)

We must express the condition that the complex quantity ϕ, as a function of the real variable defining the position of P on the line X, has positive frequency. Let us imagine† X as a finite line. (The concept of a 'positive frequency function' defined on a null line is, in fact, invariant under restricted conformal transformations of M, since these give the restricted projective group on a real null line.‡

† To be more precise, 'imagine ...' can be interpreted to mean 'make a conformal transformation on M, continuous with the identity, so that after the transformation we have ...'. Thus, we are at liberty to choose coordinates so that certain points at infinity receive finite coordinates.

‡ See Gel'fand, I. M., Graev, M. I. and Vilenkin, N. Ya. (1966). *Generalized Functions V: Integral Geometry and Representation Theory*. Academic Press, New York and London.

But this is not really needed here, since we can imagine I to remain at infinity.)

For simplicity, imagine our origin of coordinates to be situated on X. Then the real points of X have position vectors of the form ul^a, where u is a real number and l^a is a real future-pointing null vector. Along X, we now have ϕ as a complex function of the real variable u. This defines the amplitude of our plane-wave component. The wave has positive frequency if this function can be extended to a complex analytic function of the *complex* variable $u + iw$ which is regular for $w > 0$. (The regular behaviour at infinity of ϕ will be taken care of by the fact that we are really concerned with entire *compact* manifold M.) Now, the position vectors $(u + iw)l^a$, with $w > 0$, have null future-pointing imaginary parts. Therefore (cf. Section 2) they define points P of M^* which correspond in the C-picture, to lines lying in $N \cup C^+$. If $\phi(X^\alpha; P^{\alpha\beta})$ remains regular for these points, then we shall have the required analytically extended function. (The fact that ϕ is a *complex analytic* function of $u + iw$ follows from the general analytic properties of the C-picture, but this can also be seen explicitly if we go back to the definition of ϕ_0 given by (3.1), where u is allowed to be replaced by the complex variable $u + iw$). Thus, if we can construct $\phi_r(X^\alpha, Y^\alpha)$ which is regular *whenever* the line joining X and Y lies in $N \cup C^+$, we shall certainly have constructed a field of positive frequency. (In fact *all* fields of positive frequency will have this apparently more general property, but the matter will not be entered into here.)

Now let us consider a condition on f which ensures that ϕ_r is of this type. We suppose, in fact, that the points Z in $N \cup C^+$, for which $f(Z^\alpha)$ is singular, can be contained in *two disjoint closed proper subsets* S_+, S_- of $N \cup C^+$. For each line P, in C, which intersects both of $S_\pm$, we shall choose our contour so as to surround the whole of $P \cap S_+$ once in a positive sense (on the Riemann sphere S^2 which represents the points of P). Thus, the contour correspondingly surrounds the whole of $P \cap S_-$ once in a negative sense. (This can always be arranged since $P \cap S_+$ and $P \cap S_-$ are disjoint closed sets. The contour need not be a connected curve but may consist of several loops.) If either of S_+ or S_- is vacuous the situation becomes trivial and the field ϕ_r would have to be zero. I shall suppose that ϕ_r is, in fact, not identically zero. Then at least some lines in C must intersect both of $S_\pm$. In fact, if $P \subset N \cup C^+$, then P *necessarily* intersects both of S_+, S_-, since otherwise every line in $N \cup C^+$ 'sufficiently close' to P would share P's behaviour and we should have an open set in M^* on which ϕ_r vanishes. This would imply $\phi_r \equiv 0$, since ϕ_r is analytic, contrary to hypothesis.

We see that the field ϕ_r is uniquely determined, for $P \subset N \cup C^+$

(and for certain other P's also) by the division of the singularities of f between the two sets S_+, S_-. No more precise specification of the contour is required. Thus, by the preceding discussion, $\phi_r(X^\alpha, Y^\alpha)$ will be defined and analytic whenever the join of X and Y lies in $N \cup C^+$, whence ϕ_r is necessarily a field of *positive frequency*. To obtain a field of negative frequency we need only apply the identical construction but with C^- replacing C^+. We can also generalise the discussion, somewhat, if we consider functions $f(Z^\alpha)$ where the singularities need be disconnected only in C^+ rather than in $N \cup C^+$. The resulting fields ϕ_r would then not necessarily be regular at all real points P of M. This would enable us to consider non-analytic fields and distributions (Green's functions) on M as boundary values of complex analytic fields.†

It is natural to ask how general the above construction is, for positive frequency fields. It appears, in fact, to be as general as one would wish it to be, but a complete argument is at the moment lacking. To indicate something of the generality involved, however, I can exhibit some examples. Let Q_α and R_α be two right-handed twistors for which every linear combination is also right-handed. This is achieved by taking $\bar{Q}$ and $\bar{R}$ as two points of a line lying entirely in C^+ and we have

$$Q_\alpha \bar{Q}^\alpha > 0, \qquad R_\alpha \bar{R}^\alpha > 0, \qquad Q_\alpha \bar{Q}^\alpha R_\beta \bar{R}^\beta > |Q_\alpha \bar{R}^\alpha|^2 \tag{3.22}$$

The intersection of the two planes Q and R (being the 'polar line' of the line joining $\bar{Q}$, $\bar{R}$) will lie entirely in C^-. Thus $S_+ = Q \cap (N \cup C^+)$ and $S_- = R \cap (N \cup C^+)$ will be disjoint closed subsets of $N \cup C^+$ containing the relevant singularities of

$$f(Z^\alpha) \equiv (Q_\alpha Z^\alpha)^{-1}(R_\beta Z^\beta)^{-2s-1} \tag{3.23}$$

Of course (if $s > 0$) we could allow other negative powers of $Q_\alpha Z^\alpha$. Furthermore, we can form various finite linear combinations of expressions like (3.23), with differing Q's and R's. We could also take combinations of expressions like (3.23), but with polynomials in Z^α in the numerator and higher powers in the denominator. These would all yield f's which were algebraic functions of Z^α, but we could generate non-algebraic functions also satisfying the required conditions, simply by integrating together expressions like (3.23) over some (not too extensive) domain. That this yields positive frequency fields ϕ_r of considerable generality, at least, follows from the fact that in the

† See, for example, Streater, R. F. and Wightman, A. S. (1964). *PCT, Spin and Statistics and All That.* W. A. Benjamin, New York.

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limit when the inequalities (3.22) tend to equalities, the field generated by (3.23) becomes the 'Feynman propagator' for $\phi_{AB\ldots L}$ appropriate to the characteristic initial value problem.

Finally, it may be mentioned that there is an interesting topological property of such 'positive frequency' singularity sets in relation to the topology of N, as imbedded in C. I shall illustrate this explicitly, only in the case of f given by (3.23). As was mentioned in Section 2, we have $N \cong S^2 \times S^3$, $C^- \cong C^+ \cong S^2 \times E^4$. We may regard C^+ as being obtained from N by 'filling in' each S^3 with a cell E^4. The complex plane Q meets N in an S^3 of the product $S^2 \times S^3$. The intersection of Q with C^+ is an E^4 of the product $S^2 \times E^4$ which 'fills in' this S^3. Similarly, the complex plane R meets N in another S^3 of the product $S^2 \times S^3$ which is again 'filled in' by an E^4 $(= R \cap C^+)$ of the product $S^2 \times E^4$ $(= C^+)$. In fact the *pencil of planes* defined by Q and R (with twistors $Q^\alpha + \mu R^\alpha$) generates the entire 'filling in' of $S^2 \times S^3 = N$ by $S^2 \times E^4 = C^+$. Now, from the symmetry between C^+ and C^-, it follows that C^- must also 'fill in' N in exactly the same way. However, this new 'filling in' applies to an *essentially different way of expressing N as $S^2 \times S^3$* from the one given above. If we consider N as imbedded in $N \cup C^-$ rather than in $N \cup C^+$, it emerges that the sets $Q \cap N$ and $R \cap N$ are *linked* in the sense that any submanifold of C^- which spans $Q \cap N$ (i.e. whose boundary is $Q \cap N$) *must necessarily* intersect any submanifold of C^- which spans $R \cap N$. (Furthermore, neither $Q \cap N$ nor $R \cap N$ spans an E^4 in C^-.) Thus, we see that a particular topological structure of the singularity regions of our 'positive frequency' function $f(Z^\alpha)$ emerges even in the region $Z \in N$. The regions $N \cap S_+$ and $N \cap S_-$ twist around one another in a way opposite to the corresponding behaviour for a 'negative frequency' function. This is realised, in the M-picture, as a right-handed twist (for 'positive frequency' f) and a left-handed twist (for 'negative frequency' f) for the null line systems concerned.†

† The reader may be disturbed by this association of a screw-sense in space-time with the notion of 'positive frequency' for a zero rest-mass field. In the present formalism, the opposite association would have been achieved had we been concerned with a spinor field $\psi_{A'B'\ldots L'}$ rather than $\phi_{AB\ldots L}$. With half-odd spin fields this association of a particle helicity with the field is familiar. Here, this must be applied also to integral spin. Thus ϕ_{AB} describes the 'photino' (or positive helicity photon) while $\bar{\phi}_{A'B'}$ describes the 'anti-photino'. Interactions will readily convert 'photinos' into 'anti-photinos'.

The fact that the use of twistors entails an interrelation between '$+i$', 'right-handedness', and 'future-orientation' will be evident from the interpretation of the *lines* in C^+ as points of M^* with position vectors having future-pointing imaginary parts, and the interpretation of *points* of C^+ in terms of right-handed Robinson congruences.

4. *Twistors for Curved Space-time*

We have seen, in Section 3, the importance of the complex analytic structure of the C-picture in the treatment of zero rest-mass fields. On the other hand, it is (apparently) precisely this complex analytic structure which is destroyed by the presence of conformal curvature (i.e., gravitation) in the M-picture. For Kerr's theorem tells us that the Cauchy-Riemann equations for the C-picture have an interpretation as a 'shear-free' condition on null line systems in the M-picture. But when a shear-free 'bundle' of null lines (geodesics) enters a region of conformal curvature in space-time it will emerge on the other side *possessing* shear, in general. Thus, if M is conformally curved, we cannot interpret 'shear-freeness' as referring to null lines in their entirety, but only to null lines 'in the neighbourhood of a point'. We *may* think of a C-picture as referring accurately only to a 'sufficiently small' neighbourhood of a point of M, for which the conformal curvature can be neglected. But we would like, also, to be able to discuss the interconnections between different neighbourhoods, so that the real effect of conformal curvature (that is, of gravitation) on C-picture structure can be investigated. It will emerge, in fact, that what is required is not a different C-picture for each 'neighbourhood of a point' in space-time, but rather one C-picture for the whole of space-time (roughly speaking) whose 'complex analytic structure' appears to 'shift' as we move about the space-time, and which possess a (weaker) *symplectic* structure which does *not* shift. This will be the *classical* C-picture for M which we consider in this section. The passage to a quantised theory (Section 5) which this classical C-picture structure suggests will, in a certain sense, reinstate the full C-picture analyticity.

Let us suppose that M is a four-dimensional manifold with a pseudo-Riemannian metric $ds^2 = g_{ab}dx^a dx^b$, of signature $(+, -, -, -)$, and which possesses suitable global properties. These global restrictions on M will not concern us particularly here and we may, if desired, simply restrict our attention to some suitably well-behaved open submanifold of space-time. The idea will be to represent the null geodesics of M as the points of some five-dimensional manifold N. The *invariant structure* of N will be defined in terms of properties of null geodesics in M which do not refer to *particular* points on these null geodesics, but which can be read off by examining the geodesics at *any* of their points. For example, the property of a system of null geodesics that they should generate a *null hypersurface*,† will turn out

† That is, so that the tangent 3-space to the hypersurface is tangent to the light cone. This property 'propagates' in the sense that it holds globally (except for singularities) if it holds on any space-like cross-section of the hypersurface.

to represent an aspect of the invariant structure of N. On the other hand, the property of two null geodesics that they *intersect*, will not. It will emerge that the invariant structure of N will be most easily describable if we regard N as a submanifold of a six-real-dimensional manifold C. In this sense, the classical C-picture will have value for the description even of conformally curved space-time.

The discussion will be given here in terms of certain idealised space-times in the first instance, namely plane waves—or, more generally, plane-fronted waves (Brinkmann, 1923; Robinson, 1958)—for which the amplitudes can be given by Dirac delta functions. The idea will then be to regard the effect on the C-picture, of a general region of M-picture curvature, as a (non-linear) composition of effects of such plane-fronted waves.

The general plane-fronted wave has a metric which can be put into the form

$$ds^2 = 2(du + R(v, \zeta, \bar{\zeta})\, dv)\, dv - 2d\zeta\, d\bar{\zeta} \tag{4.1}$$

(R being real). The non-vanishing curvature tensor components for the metric (4.1) are defined by

$$\partial^2 R/\partial\zeta^2, \qquad \partial^2 R/\partial\zeta\,\partial\bar{\zeta}, \qquad \partial^2 R/\partial\bar{\zeta}^2 \tag{4.2}$$

The metric satisfies Einstein's vacuum equations (and so represents a plane-fronted purely gravitational wave) if

$$\partial^2 R/\partial\zeta\,\partial\bar{\zeta} = 0 \tag{4.3}$$

but we shall not be concerned with implications of (4.3) here. If (4.3) is *not* imposed, then the metric (4.1) covers the more general situation of a combined plane-fronted gravitational-electromagnetic-neutrino wave.

If $R = 0$ over some range of v, then we have a region of Minkowski space-time [compare (2.3)]. If R is non-zero only within some finite range of v: $v_1 < v < v_2$, then we have the situation of a 'sandwich wave'. In the idealized situation where the wave is allowed to become infinitesimal in duration (say, $v_1, v_2 \to 0$) while still producing a non-zero resultant effect, the function R becomes a delta function in v:

$$R(v, \zeta, \bar{\zeta}) \equiv \delta(v)\, r(\zeta, \bar{\zeta}) \tag{4.4}$$

With the substitution (4.4), the form of metric (4.1) does not satisfy the conditions normally required (Bruhat, 1959) for a space-time with delta functions in the curvature, since we here have a delta function in the *metric* tensor components. Under normal circumstances this would lead to non-allowable *products* of delta functions in addition to derivatives of delta functions in the curvature. However, in the present situation, the space-time represented by (4.1), (4.4) is still allowable. It is, in fact, possible to make a coordinate substitution so that the new metric tensor components are C^0 functions of the coordinates. Then the fact that this leads only to a simple delta function (C^{-2}) type of curvature becomes more obvious. But for our purposes the form (4.1) is more satisfactory since it leaves the flat portions of the space-time in the required Minkowski form (2.3). The (delta function) curvature components are then obtained from the substitution of (4.4) into (4.2).

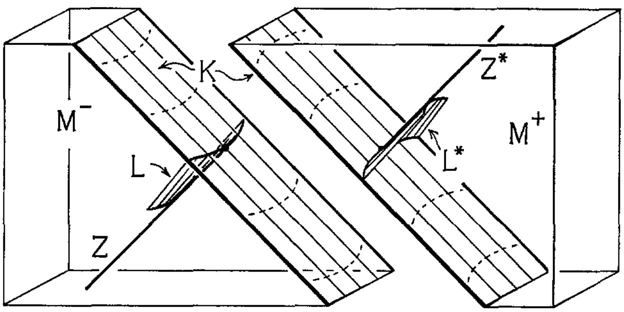

Figure 5.—The two Minkowski half-spaces are joined in a 'warped' fashion along the null hyperplane K. The continuation Z^* of the null line Z can be defined in terms of the null hypersurface L, which continues as L^*.

We can describe the resulting manifold in the following 'scissors and paste' terms. We divide ordinary Minkowski space-time (metric $ds^2 = 2du\,dv - 2d\zeta\,d\bar{\zeta}$) into two portions M^-, M^+ by the removal of the null hyperplane $v = 0$. Thus M^- is given by the portion $v < 0$ and M^+ by $v > 0$. We wish to join M^- and M^+ together again, inserting a null hyperplane K as their common boundary, but in such a way that the imbedding of K in each of $M^\pm$ appears 'warped' as viewed from the other (Fig. 5). More specifically, $M^- \cup K$ and $M^+ \cup K$ each have the normal Minkowski metric (2.3), but the two halves are joined in a way not consistent with a four-dimensional Minkowskian metric structure at K. The three-dimensional metric induced on K by its imbedding in each of the two halves is the same, however. (A somewhat analogous two-dimensional example is obtained if we imagine two ordinary cones joined base to base. The two surfaces are intrinsically flat, but a delta function in the curvature resides along the edge at which the cones are joined.) For convenience, we use coordinates u, v, ζ in M^- and u^*, v^*, ζ^* in M^+, the entire manifold M being defined by:

$$M^- \cup K: \quad ds^2 = 2du\,dv - 2d\zeta\,d\bar{\zeta}$$

$$(v \leqslant 0 \text{ with } u, \zeta \text{ unrestricted; } u \text{ real})$$

$$M^+ \cup K: \quad ds^2 = 2du^*\,dv^* - 2d\zeta^*\,d\bar{\zeta}^* \tag{4.5}$$

$$(v^* \geqslant 0 \text{ with } u^*, \zeta^* \text{ unrestricted; } u^* \text{ real})$$

$$K: \quad v^* = v = 0, \qquad \zeta^* = \zeta, \qquad u^* = u - r(\zeta, \bar{\zeta})$$

Thus, the generators of K are 'shunted down' by an amount $r(\zeta,\bar{\zeta})$ when we pass from M^- to M^+. Equations (4.5) are just another way of expressing what is meant by the metric (4.1) when the substitution (4.4) is made.

The manifold M possesses two regions M^- and M^+ which are exactly flat. Thus, we can construct a C-picture in terms of either of these regions. A null line in M^- can be given twistor coordinates Z^α according to the scheme (2.8); similarly a null line in M^+ can be given twistor coordinates $Z^{*\alpha}$ according to the 'starred' version of (2.8). But any null line in M^- which intersects K (i.e., which is not parallel to the null direction in K) will emerge as the *uniquely defined* null line, in M^+, for which the two portions constitute a 'null geodesic' in M. We may think of such a 'null geodesic' as resulting when a limiting process is applied to null geodesics for spaces (4.1), as the function $R(v,\zeta,\bar{\zeta})$ approaches the form (4.4). More conveniently, there is, however, a direct geometrical construction of these 'null geodesics'. This arises owing to the fact that if null geodesics generate a null hypersurface in one portion of a space-time manifold, then they must continue to generate a null hypersurface in any other portion of the manifold. We must *retain* this property for the 'null geodesics' in M (as follows from the above limit construction). Thus, if the null line Z in $M^- \cup K$ belongs to a null hypersurface L in $M^- \cup K$, the emergent null line Z^* in $M^+ \cup K$ must belong to a corresponding null hypersurface L^* in $M^+ \cup K$. We require that: $Z \cap K = Z^* \cap K$ and $L \cap K = L^* \cap K$. This serves to define Z^* uniquely in terms of Z, because $Z^* \cap K$ fixes a *point* on Z^* while $L^* \cap K$ defines a tangent 2-plane element at $Z^* \cap K$ which must be orthogonal to Z^*, thereby fixing the *direction* of Z^*.

Let us see this explicitly in terms of (4.5). A twistor Z^α, describing the null line Z according to (2.6), (2.8), satisfies

$$Z^0\,d\bar{\zeta} = Z^1\,du, \qquad Z^0\,dv = Z^1\,d\zeta \tag{4.6}$$

$$Z^2 = i\zeta Z^1, \qquad Z^3 = i\bar{\zeta}Z^0 - iuZ^1 \tag{4.7}$$

(Here, the point P is chosen to be $Z \cap K$, so that $v = 0$.) Similarly, the starred versions of (4.6), (4.7) will also hold. In the case of (4.7), employing (4.5) we get

$$Z^{*2} = i\zeta Z^{*1}, \qquad Z^{*3} = i\bar{\zeta}Z^{*0} - i(u-r)Z^{*1} \tag{4.8}$$

In order to write the starred version of (4.6) in terms of $du, dv, d\zeta$ (i.e., to find the direction of Z^* given Z) we need to use the fact that Z^* is orthogonal to the same vectors lying within K, at $Z^* \cap K$, as is Z. Denoting a direction at $Z \cap K$ by $\delta u : \delta v : \delta\zeta$ in the u, v, ζ system, we have $\delta v = 0$ if the direction is to lie within K. For the direction to be orthogonal to that of Z, we have

$$\delta u\, dv + 0 \,.\, du = \delta\zeta\, d\bar{\zeta} + \delta\bar{\zeta}\, d\zeta \tag{4.9}$$

whence, by (4.6),

$$\delta u = \delta\zeta \overline{Z^0}/\overline{Z^1} + \delta\bar{\zeta} Z^0/Z^1 \tag{4.10}$$

The starred version of this yields, with (4.5),

$$\delta u - \frac{\partial r}{\partial \zeta}\delta\zeta - \frac{\partial r}{\partial \bar{\zeta}}\delta\bar{\zeta} = \delta\zeta \overline{Z^{*0}}/\overline{Z^{*1}} + \delta\bar{\zeta} Z^{*0}/Z^{*1} \tag{4.11}$$

Equations (4.10) and (4.11) must represent identical conditions on $\delta u : \delta\zeta : \delta\bar{\zeta}$ since they must give the same 2-plane element. Hence,

$$Z^0 : Z^1 = Z^{*0} + Z^{*1}\frac{\partial r}{\partial \bar{\zeta}} : Z^{*1} \tag{4.12}$$

Equations (4.7), (4.8) and (4.12) define the ratios of the $Z^{*\alpha}$ components in terms of the ratios of the Z^α components, by elimination of ζ and u. With the most convenient choice of scale factor, we can set

$$Z^{*0} = Z^0 - Z^1\frac{\partial r}{\partial \bar{\zeta}}, \quad Z^{*1} = Z^1$$
$$Z^{*2} = Z^2, \qquad Z^{*3} = Z^3 + iZ^1\left(r - \bar{\zeta}\frac{\partial r}{\partial \bar{\zeta}}\right) \tag{4.13}$$

where $\zeta = -iZ^2/Z^1$. Setting

$$h(Z^\alpha, \bar{Z}_\alpha) \equiv |Z^1|^2 r \tag{4.14}$$

we can write (4.13) comprehensively as

$$Z^{*\alpha} = Z^\alpha + i\frac{\partial h}{\partial \bar{Z}_\alpha} \tag{4.15}$$

The transformation (4.15) has been derived for the effect due to a plane-fronted impulsive wave situated along the particular null hyperplane $v = 0$. But because of its twistor form, (4.15) will clearly apply (with suitable h) to *any* impulsive plane-fronted wave. The function $h(Z^\alpha, \bar{Z}_\alpha)$ of (4.14) is special in that it involves only the coordinates Z^2, Z^1 and their complex conjugates. Putting

$$A^\alpha = (1, 0, 0, 0) \qquad \text{and} \qquad B^\alpha = (0, 0, 0, 1) \tag{4.16}$$

we can express this as the fact that h is a function of

$$\bar{A}_\alpha Z^\alpha, \bar{B}_\alpha Z^\alpha; \qquad A^\alpha \bar{Z}_\alpha, B^\alpha \bar{Z}_\alpha \tag{4.17}$$

only. More generally, we could allow h to be any *real differentiable function* of variables of the type (4.17), so that h is *homogeneous of degree unity in* Z^α *and also in* $\bar{Z}_\alpha$, where we allow A^α and B^α to *any* pair of fixed null twistors satisfying $A^\alpha \bar{B}_\alpha = 0$. Note that $\partial h / \partial \bar{Z}_\alpha$ is a linear combination of A^α and B^α whence

$$\frac{\partial h}{\partial \bar{Z}_\alpha} \frac{\partial h}{\partial Z^\alpha} = 0 \tag{4.18}$$

Note, also

$$\frac{\partial^2 h}{\partial \bar{Z}_\alpha \partial Z^\alpha} = 0 \tag{4.19}$$

For the particular case considered in detail above, it is the hyperplane $v = 0$ which is generated by the null lines given by $B^\alpha + i\bar{\zeta} A^\alpha$. In the general case, these null lines would be just the generators of the null cone containing two intersecting null lines A and B [cf. (2.10)]. In the above case (4.16), A is at infinity so that the cone becomes a hyperplane. In fact, unless one of the generators of the cone *is* at infinity, (4.15) does not represent a pure impulse wave in a pseudo-Riemannian space-time but only in a *conformal* space-time. We shall ignore this distinction for the present purposes. There is, however, one special situation which is of particular note, namely when *both* A and B are at infinity, so that the null cone they define becomes the null cone at infinity itself. In this case we may regard the transformation involved in (4.5) as yielding a *supertranslation of the Bondi-Metzner-Sachs group*† and (4.15) is its twistor equivalent.

Observe that (4.18) (together with reality: $h = \bar{h}$; and homogeneity: $Z^\alpha \partial h / \partial Z^\alpha = h$) implies that

$$Z^{*\alpha} \overline{Z^*}_\alpha = Z^\alpha \bar{Z}_\alpha \tag{4.20}$$

so that the scaling we chose for $Z^{*\alpha}$, in order to arrive at (4.15), in fact preserves the twistor 'norm'. However, (4.15) does *not* preserve the twistor scalar product $Z^\alpha \bar{Y}_\alpha$. In particular, if $Z^\alpha \bar{Y}_\alpha = 0$ (so the null lines Z and Y, in M^- belong to a portion of a null cone), then in general $Z^{*\alpha} \overline{Y^*}_\alpha \neq 0$ (so that the emergent Z^*, Y^* in M^+ do not belong

† See Sachs, R. K. (1962). *Physical Review*, **128**, 2851.

to a portion of a null cone). This is closely related to the non-analyticity, in Z^α, of (4.15), since it illustrates that a (shear-free) null cone picks up shear, in general, as it passes through an impulsive plane-fronted wave. (The only cases of a transformation (4.15) yielding a complex analytic twistor transformation would be given when h is bilinear in Z^α, $\bar{Z}_\alpha$. In this case the conformal curvature in the impulsive wave would vanish and M would be conformally flat.)

We note the important fact that whereas we originally defined the transformation $Z^\alpha \rightarrow Z^{*\alpha}$ for *null* twistors only (since the discussion was given in terms of null lines) we were led to the transformation (4.15) which applies equally to *non-null* twistors. (We may, perhaps, think of (4.19) as defining h away from N once the values of h on N are given.) Thus, the identity of a twistor can apparently be maintained even as we pass through a region of conformal curvature. The *geometrical* significance of a non-null twistor becomes altered, however. The representation of such a twistor by a Robinson congruence (cf. Section 2) cannot be maintained in the presence of conformal curvature, since this representation is based on the condition $Y^\alpha \bar{Z}_\alpha = 0$, for a null line Y to belong to the congruence $\bar{Z}$. Nevertheless the entire C-picture for M does seem to retain a significance. We may think of a single point Z in the C-picture as being assigned twistor coordinates according to two different (non-analytically related) schemes, namely that which assigns the coordinates Z^α, and that which assigns $Z^{*\alpha}$. Thus we have two *different* complex analytic structures for C and two different scalar products, depending upon whether we view it from M^- or from M^+.

The space-time M that we have just been considering is, of course, especially simple. We cannot expect to get such a complete twistor structure for a space-time M which is, perhaps, everywhere conformally curved. However, the purpose here is somewhat different. We may imagine that the effect of a region of general conformal curvature can, in some way, be *composed* of effects like that produced by an impulsive plane-fronted wave. We are thus led to consider the group $\mathscr{T}$ of transformations of twistor coordinates, which is generated by the transformations like (4.15). The invariant structure of C will then be precisely that which is left invariant by $\mathscr{T}$.

It is convenient to consider the infinitesimal transformations of the form (4.15). Denoting the infinitesimal change in the twistor coordinates Z^α by δZ^α, we can write

$$\delta Z^\alpha = i\frac{\partial H}{\partial \bar{Z}_\alpha} \qquad (4.21)$$

where

$$H = H(Z^\alpha, \bar{Z}_\alpha) \tag{4.22}$$

is real and separately homogeneous of degree unity in Z^α and in $\bar{Z}_\alpha$:

$$H = \overline{H} = Z^\alpha \frac{\partial H}{\partial Z^\alpha} = \bar{Z}_\alpha \frac{\partial H}{\partial \bar{Z}_\alpha} \tag{4.23}$$

Thus, we have

$$\delta \bar{Z}_\alpha = -i \frac{\partial H}{\partial Z^\alpha} \tag{4.24}$$

It follows now that

$$\delta(Z^\alpha \bar{Z}_\alpha) = 0 \tag{4.25}$$

so that the invariance of the twistor norm follows without any other conditions on H. We need not assume that H is of the special form that was required for h, namely of being a function of variables (4.17), with its consequence (4.18). Indeed, neither (4.18) nor (4.19) can be expected to apply to a general H defining an infinitesimal element of $\mathscr{T}$. This is because the *Poisson brackets*:

$$[\psi, \chi] \equiv i \frac{\partial \psi}{\partial Z^\alpha} \frac{\partial \chi}{\partial \bar{Z}_\alpha} - i \frac{\partial \psi}{\partial \bar{Z}_\alpha} \frac{\partial \chi}{\partial Z^\alpha} \tag{4.25a}$$

do not preserve (4.18) or (4.19).

The Poisson brackets $[H, G]$ define the commutator of the two infinitesimal transformations defined by H and by G. Note that $[H, G]$ satisfies the reality and homogeneity conditions (4.23) provided H and G both satisfy these conditions. The same applies to the sum $H + G$. Thus, these operations define a Lie algebra $\mathscr{L}$. It is clear that $\mathscr{L}$ will contain the Lie algebra of infinitesimal elements of $\mathscr{T}$. (Very possibly $\mathscr{L}$ is no larger than this.) In any case it will follow that any structure on C which is invariant under $\mathscr{L}$ will also be invariant under $\mathscr{T}$, i.e., be part of the *invariant structure* of C. The converse is an open question at present.

Note that we can write (4.21), (4.24) as

$$\delta Z^\alpha = [Z^\alpha, H], \qquad \delta \bar{Z}_\alpha = [\bar{Z}_\alpha, H] \tag{4.26}$$

and, more generally, we have

$$\delta \psi = [\psi, H] \tag{4.27}$$

for any function ψ of Z^α, $\bar{Z}_\alpha$. Thus, if $[\psi, H] = 0$ for all H satisfying (4.23), then ψ is part of the invariant structure of C. By (4.25), a particular case would be $\psi = Z^\alpha \bar{Z}_\alpha$. To obtain more of the invariant structure of C, we must consider 'tensor fields' on C. In particular, there are certain invariant *differential forms*† on C. Noting that, by (4.21),

$$\delta(dZ^\alpha) = d\left(i\frac{\partial H}{\partial \bar{Z}_\alpha}\right) = i\frac{\partial^2 H}{\partial Z^\beta \partial \bar{Z}_\alpha} dZ^\beta + i\frac{\partial^2 H}{\partial \bar{Z}_\beta \partial \bar{Z}_\alpha} d\bar{Z}_\beta \tag{4.28}$$

and using the fact that $\partial H/\partial Z^\beta$ and $\partial H/\partial \bar{Z}_\beta$ are respectively homogeneous of degrees one and zero in $\bar{Z}_\alpha$, we obtain

$$\delta(\bar{Z}_\alpha dZ^\alpha) = 0 \tag{4.29}$$

Thus $\bar{Z}_\alpha dZ^\alpha$ is part of the invariant structure of C and, hence, so also is its exterior derivative $d(\bar{Z}_\alpha dZ^\alpha) = d\bar{Z}_\alpha \wedge dZ^\alpha = 0$, i.e.

$$\delta(d\bar{Z}_\alpha \wedge dZ^\alpha) = 0 \tag{4.30}$$

The quantity $d\bar{Z}_\alpha \wedge dZ^\alpha$ is the usual invariant surface element associated with the 'Hamiltonian' equations (4.21), (4.24). By taking exterior products of these forms, higher degree invariant quantities can be (rather trivially) generated, e.g. $Z^\alpha d\bar{Z}_\alpha \wedge dZ^\beta \wedge d\bar{Z}_\beta$; $\bar{Z}_\alpha Z^{(\alpha} dZ^{\beta)} \wedge d\bar{Z}_\beta$. The 7-form $Z^\alpha d\bar{Z}_\alpha \wedge dZ^\beta \wedge d\bar{Z}_\beta \wedge dZ^\gamma \wedge d\bar{Z}_\gamma \wedge dZ^\mu \wedge d\bar{Z}_\mu$ that can be built in this way is dual to $Z^\alpha \partial/\partial Z^\alpha$. The invariance of this operator follows more directly from

$$\left[Z^\alpha \frac{\partial \psi}{\partial Z^\alpha}, H\right] = Z^\alpha \frac{\partial}{\partial Z^\alpha}[\psi, H] \tag{4.31}$$

which we can interpret as

$$\delta\left(Z^\alpha \frac{\partial}{\partial Z^\alpha}\right) \equiv \left(Z^\alpha \frac{\partial}{\partial Z^\alpha}\right)\delta \tag{4.32}$$

This is the condition for invariance of the operator $Z^\alpha \partial/\partial Z^\alpha$ under δ.

The most important quantities belonging to the invariant structure of C can be collected together as follows:

$$\bar{Z}^\alpha Z_\alpha \tag{4.33}$$

$$Z^\alpha d\bar{Z}_\alpha, \qquad \bar{Z}_\alpha dZ^\alpha \tag{4.34}$$

$$dZ^\alpha \wedge d\bar{Z}_\alpha \tag{4.35}$$

$$Z^\alpha \frac{\partial}{\partial Z^\alpha}, \qquad \bar{Z}_\alpha \frac{\partial}{\partial \bar{Z}_\alpha} \tag{4.36}$$

It is of interest that all these quantities have *some* direct significance in the M-picture. It is the vanishing of (4.33) which states that the twistor Z^α represents a *real null line* Z in M. The division of C into the

† See footnote on page 92.

three parts C^-, N, C^+ is invariant, being given by $Z^\alpha \bar{Z}_\alpha > 0, = 0, < 0$, respectively. However, the points of C^+ and C^- do not appear to have any very direct invariant geometrical interpretation in M. As for (4.34), we may think of Z^α and $Z^\alpha + dZ^\alpha$ as defining two infinitesimally neighbouring points Z and Z' in C. If Z and Z' are both on N, then in the M-picture, Z and Z' are infinitesimally neighbouring null lines and we have $Z^\alpha \bar{Z}_\alpha = 0$ and $Z^\alpha d\bar{Z}_\alpha + dZ^\alpha \bar{Z}_\alpha = 0$ (neglecting quantities of second order in dZ^α). Thus, $Z^\alpha d\bar{Z}_\alpha$ is pure imaginary. Now, the imaginary part of $Z^\alpha d\bar{Z}_\alpha$ is *positive* or *negative* according as the null line Z' lies just to the *future* or just to the *past* of Z. If $Z^\alpha d\bar{Z}_\alpha = 0$, then a *null hypersurface* can contain both Z and Z' as neighbouring generators. Now consider (4.35). We may think of Z^α, $Z^\alpha + dZ^\alpha$ and $Z^\alpha + d'Z^\alpha$ as defining three neighbouring null lines Z, Z' and Z'' in M provided these three twistors are all null. Let us suppose that this is so, and also that any pair of Z, Z', Z'' can belong to a null hypersurface, i.e. $Z^\alpha \bar{Z}_\alpha = 0$, $Z^\alpha d\bar{Z}_\alpha = 0$, $Z^\alpha d'\bar{Z}_\alpha = 0$. Then the condition that all three of Z, Z', Z'' can belong to one null hypersurface (i.e. that there be no net rotation of Z, Z', Z'' about one another) is $dZ^\alpha d'\bar{Z}_\alpha - d'Z^\alpha d\bar{Z}_\alpha = 0$, which in the notation of differential forms reads $dZ^\alpha \wedge d\bar{Z}_\alpha = 0$. The form $dZ^\alpha \wedge d\bar{Z}_\alpha$ defines a *symplectic* structure on C. Finally, the invariance of (4.36) is a necessary prerequisite for the C-picture to make geometrical sense at all. For, we have tacitly assumed that it is legitimate to think of C as a six-real-dimensional manifold, the points of which are defined by the *ratios* of the complex coordinates Z^α. The fact that the relation between Z^α and λZ^α is preserved under any transformation in $\mathscr{L}$ is implicit in the fact that the (degree of) homogeneity in Z^α of a function $\psi(Z^\alpha, \bar{Z}_\alpha)$ is preserved under $\mathscr{L}$. The functions homogeneous in Z^α are simply the eigenfunctions of the invariant operator $Z^\alpha \partial/\partial Z^\alpha$.

The above discussion has been centered on considerations of infinitesimal transformations. We may also consider finite transformations $Z^\alpha \to Z^{*\alpha}$ belonging to $\mathscr{T}$. For such transformations we must expect that the expression of the Poisson brackets in terms of Z^α, $\bar{Z}_\alpha$ will hold also for $Z^{*\alpha}$, $\overline{Z^*}_\alpha$. We have

$$[Z^\alpha, Z^\beta] = 0 = [\bar{Z}_\alpha, \bar{Z}_\beta], \qquad [Z^\alpha, \bar{Z}_\beta] = i\delta_\beta{}^\alpha \tag{4.37}$$

Thus also

$$[Z^{*\alpha}, Z^{*\beta}] = 0 = [\overline{Z^*}_\alpha, \overline{Z^*}_\beta], \qquad [Z^{*\alpha}, \overline{Z^*}_\beta] = i\delta_\beta{}^\alpha \tag{4.38}$$

These equations can be stated

$$\frac{\partial Z^{*\alpha}}{\partial Z^\beta} = \frac{\partial \bar{Z}_\beta}{\partial \overline{Z^*}_\alpha}, \qquad \frac{\partial Z^{*\alpha}}{\partial \bar{Z}_\beta} = -\frac{\partial Z^\beta}{\partial \bar{Z}_\alpha} \tag{4.39}$$

or, equivalently

$$dZ^{*\alpha} \wedge d\overline{Z^{*}}_{\alpha} = dZ^{\alpha} \wedge d\bar{Z}_{\alpha} \tag{4.40}$$

If, in addition, we assume that $Z^{*\alpha}$ is homogeneous of degree one in Z^{α} and of degree zero in $\bar{Z}_{\alpha}$, i.e.

$$Z^{*\alpha}\frac{\partial}{\partial Z^{*\alpha}} \equiv Z^{\alpha}\frac{\partial}{\partial Z^{\alpha}} \tag{4.41}$$

then it follows that

$$Z^{*\alpha}\,\overline{Z^{*}}_{\alpha} = Z^{\alpha}\,\bar{Z}_{\alpha}, \qquad Z^{*\alpha}\,d\overline{Z^{*}}_{\alpha} = Z^{\alpha}\,d\bar{Z}_{\alpha} \tag{4.42}$$

also.

To sum up, the invariant structure obtained here for the C-picture is defined by a symplectic structure given by $dZ^{\alpha} \wedge d\bar{Z}_{\alpha}$, where the 'homogeneity' operator $Z^{\alpha}\partial/\partial Z^{\alpha}$ is also invariant. Equivalently, $Z^{\alpha}\bar{Z}_{\alpha}$ and $Z^{\alpha}d\bar{Z}_{\alpha}$ are invariant. If we restrict our attention to null twistors, i.e. to N, this structure describes a (conformally invariant) geometry of null geodesics in M, which heeds only the null geodesics as a whole and does not refer to points on these null geodesics. The non-null twistors appear to play a 'catalytic' role in simplifying the description of the geometry of N. The structure so obtained for N (and for C) is of a 'universal' nature; that is to say, it does not reflect the local metric (or conformal) structure of M in any way. (We can see this by removing a portion of M and joining it smoothly on to another space-time manifold, e.g. to flat space-time. The invariant structure of C does not change.) To represent local structure of M—and, in particular, its points—we would have to refer to some additional (generally non-local) structure for N over and above its invariant structure.

5. *A Hilbert Space*

The nature of the twistor transformations induced by the presence of (conformal) curvature in the M-picture strongly suggests that, in the passage to a quantum theory, we should identify a quantum operator $\bar{Z}_{\alpha}$ with some multiple of an operator $\partial/\partial Z^{\alpha}$. In order to realise such an identification, we must obtain the vectors of the appropriate Hilbert space on which these operators are to act. It is a remarkable fact that one such space is *already* at hand, having been previously required for the twistor description of zero rest-mass fields given in Section 3. This is the space of the analytic functions $f(Z^{\alpha})$ which are employed in the contour integrals (3.10). In order to get a Hilbert space, however, we shall require a definition of norm (or of scalar product) for these functions. I shall not enter into all the details involved with this here, some of which involve topological questions concerning the singularity sets, but merely give the *formal* expression for the Hilbert space scalar product. These detailed matters will be discussed elsewhere.

Let $f(Z^\alpha)$, $g(Z^\alpha)$ be two functions of the complex variables Z^0, Z^1, Z^2, Z^3 which are homogeneous of degree $-2s-2$ and analytic in suitable domains. For example, we could choose domains of analyticity of the type considered towards the end of Section 3, where the regions of $N \cup C^+$ which were excluded, were two disjoint closed sets (each of which intersected every line in $N \cup C^+$). The scalar product is then

$$\langle g|f\rangle = ik\Gamma(2-2s)\oint \bar{g}(W_\alpha)f(Z^\alpha)(W_\beta Z^\beta)^{2s-2}\mathscr{W}\wedge\mathscr{Z}. \tag{5.1}$$

where k is a real numerical constant, $\mathscr{W}$ and $\mathscr{Z}$ are differentials given by

$$\mathscr{W} = \tfrac{1}{6}\epsilon^{\alpha\beta\gamma\delta}\, W_\alpha dW_\beta \wedge dW_\gamma \wedge dW_\delta \tag{5.2}$$

$$\mathscr{Z} = \tfrac{1}{6}\epsilon_{\alpha\beta\gamma\delta}\, Z^\alpha dZ^\beta \wedge dZ^\gamma \wedge dZ^\delta \tag{5.3}$$

(the ϵ's being the usual Levi-Civita symbols), and where the (six-dimensional) region of integration (contour) is compact, and surrounds the singularities of f and g, and the region $W_\beta Z^\beta = 0$ in a suitable way. When $s = 1, 1\frac{1}{2}, 2, \ldots$ the expression (5.1) is not defined as it stands, the value of the integral being zero and the multiplying factor infinite. In these cases we can, however, assign a meaning to (5.1) as the result of a *limit* process applied to s, where s is taken as a continuous variable. This will be discussed a little more shortly.

We have to establish that (5.1) is invariant under homologous deformations of the contour over regions of analyticity of the integrand. Thus† we must verify that the exterior derivative of the expression after the integral sign vanishes. As a lemma towards achieving this, consider any function $p(Z^\alpha; u_1, \ldots, u_h)$ which is analytic and homogeneous of degree -4 in the Z^α and which depends differentiably on the parameters $u_1, \ldots, u_h$. Then

$$d(p\mathscr{Z}) = \frac{\partial p}{\partial u_i} du_i \wedge \mathscr{Z} \tag{5.4}$$

† For an account of exterior calculus, see, for example, Hodge, W. V. D. (1952). *Theory and Applications of Harmonic Integrals*. Cambridge University Press; Flanders, H. (1963). *Differential Forms, with Applications to the Physical Sciences*. Academic Press, New York and London.

To demonstrate the validity of (5.4), we must show that the terms involving derivatives of the Z^α cancel out. We have

$$d(p\mathscr{Z}) = p\,d\mathscr{Z} + \frac{\partial p}{\partial Z^\mu} dZ^\mu \wedge \mathscr{Z} + \frac{\partial p}{\partial u_i} du_i \wedge \mathscr{Z} \tag{5.5}$$

and

$$\begin{aligned}\frac{\partial p}{\partial Z^\mu} dZ^\mu \wedge \mathscr{Z} &= \frac{1}{6}\frac{\partial p}{\partial Z^\mu} Z^\alpha \epsilon_{\alpha\beta\gamma\delta} dZ^\mu \wedge dZ^\beta \wedge dZ^\gamma \wedge dZ^\delta \\ &= \frac{1}{4!6}\frac{\partial p}{\partial Z^\mu} Z^\alpha \epsilon_{\alpha\beta\gamma\delta}\, \epsilon^{\mu\beta\gamma\delta}\, \epsilon_{\lambda\rho\sigma\tau} dZ^\lambda \wedge dZ^\rho \wedge dZ^\sigma \wedge dZ^\tau \qquad (5.6)\\ &= \frac{1}{4} Z^\mu \frac{\partial p}{\partial Z^\mu}\cdot\frac{1}{6}\epsilon_{\lambda\rho\sigma\tau} dZ^\lambda \wedge dZ^\rho \wedge dZ^\sigma \wedge dZ^\tau\end{aligned}$$

by the skew-symmetry of $dZ^\mu \wedge dZ^\beta \wedge dZ^\gamma \wedge dZ^\delta$ and the properties of the ϵ's. Now, by the homogeneity of p we have

$$\tfrac{1}{4} Z^\mu \frac{\partial p}{\partial Z^\mu} = -p \tag{5.7}$$

Furthermore,

$$d\mathscr{Z} = \tfrac{1}{6}\epsilon_{\lambda\rho\sigma\tau} dZ^\lambda \wedge dZ^\rho \wedge dZ^\sigma \wedge dZ^\tau \tag{5.8}$$

Combining (5.5), (5.6), (5.7) and (5.8), we obtain (5.4) as required.

The expression $q = \bar{g}(W_\alpha) f(Z^\alpha)(W_\beta Z^\beta)^{2s-2}$ which occurs in (5.1) is analytic and homogeneous of degree -4 in Z^α and also in W_α. If we apply (5.4), with the coefficients of the 4-form $q\mathscr{W}$ in place of p, we see that $d(q\mathscr{W} \wedge \mathscr{Z})$ contains no term of degree four in the dZ's. Similarly, $d(q\mathscr{W} \wedge \mathscr{Z})$ contains no term of degree four in the dW's. Hence $d(q\mathscr{W} \wedge \mathscr{Z}) = 0$, as required. This shows that the integral (5.1) does not depend on the exact position of the contour, but only on its homology class in the region of analyticity of q. It is the singularity regions of $f(Z^\alpha)$, $\bar{g}(W_\alpha)$ and $(Z^\alpha W_\alpha)^{-2+2s}$ which must prevent the contour from being homologous to zero. Otherwise the integral in (5.1) would vanish. Something of this nature is, of course, to be expected, since it is the *separation of singularities* of f, in (3.10), which gives rise to a zero rest-mass field.

It may be verified directly that the expression (5.1) arises from precisely the usual definition of scalar product for zero rest-mass fields, in the cases $s = 0$, $s = \frac{1}{2}$. (The argument will be given elsewhere.) The cases $s = 1, 1\frac{1}{2}, 2, \ldots$ are less straightforward. Here the factor $(Z^\rho W_\rho)^{-2+2s}$ is no longer singular at $Z^\rho W_\rho = 0$, so the contour can now be deformed across this region. In fact, for the type of field considered here, the contour can always be deformed to a point, so that the integral in (5.1) vanishes. This is compensated by the pole in

$\Gamma(2-2s)$. To obtain the meaning of (5.1) in these circumstances, we may resort to a limiting argument. If f and g are mutiplied by suitable factors homogeneous of degree ϵ, and $\Gamma(2-2s)(Z^\mu W_\mu)^{-2+2s}$ is replaced by $\Gamma(2-2s+\epsilon)(Z^\mu W_\mu)^{-2+2s-\epsilon}$, then (5.1) becomes well-defined. Allowing ϵ to tend to zero and the multiplying factors to tend to unity, we can obtain a finite meaning for (5.1) when $s = 1, 1\frac{1}{2}, 2, \ldots$ also.

Let us see this explicitly, for a particular choice of multiplying factor. Choose A_α, B^β and the contour suitably, so that the contour avoids the region $A_\alpha Z^\alpha B^\beta W_\beta = 0$. (This appears to be possible under 'normal' circumstances. For more exotic functions f and g it might be necessary to choose more complicated multiplying factors.) Then

$$\begin{aligned}\langle g|f\rangle &= \lim_{\epsilon\to 0}\{ik\Gamma(2-2s+\epsilon)\times \\ &\quad\times \oint \bar{g}(W_\alpha)(B^\beta W_\beta)^\epsilon f(Z^\alpha)(A_\gamma Z^\gamma)^\epsilon (Z^\mu W_\mu)^{2-2s-\epsilon}\mathscr{W}\wedge\mathscr{Z}\} \\ &= \frac{ik(-1)^{2s-2}}{(2s-2)!}\times \\ &\quad\times \oint \bar{g}(W_\alpha)f(Z^\alpha)(Z^\mu W_\mu)^{2s-2}\ln\left\{\frac{B^\beta W_\beta A_\gamma Z^\gamma}{Z^\rho W_\rho}\right\}\mathscr{W}\wedge\mathscr{Z}\end{aligned} \tag{5.9}$$

by l'Hospital's rule, provided $s = 1, 1\frac{1}{2}, 2, \ldots$. The fact that (5.9) does not change as A_α or B^α are varied is again a consequence of the vanishing of the integral in (5.1) (but with a slightly modified function in place of f or g), as is readily verified. With this interpretation [e.g. (5.9)] for the scalar product, it can be shown that also in the cases $s = 1, 1\frac{1}{2}, 2, \ldots$ the expression (5.1) agrees with the usual definition in terms of fields. (The argument will be given elsewhere.) The more complicated behaviour for $s = 1, 1\frac{1}{2}, 2, \ldots$, than for $s = 0, \frac{1}{2}$ seems to be related to the fact that a similar increase in complication occurs in the usual formalism at this point owing to the fact that the number operator becomes non-local (or involves potentials) when $s > \frac{1}{2}$. It is also related to the existence of conserved integrals of the field (charge for $s = 1$; mass, momentum and angular momentum for $s = 2$) as we shall see shortly. Note, particularly, that the scalar product (5.1) is *conformally invariant*† (and *a fortiori* Poincaré invariant) because of twistor covariance. The definition can also be applied to *non-half-integral* spin s. The positive definiteness of (5.1) (for positive frequency fields) will be discussed elsewhere, as will the *details* of the definition of the Hilbert space (e.g. two different functions f may correspond to the same Hilbert space vector).

† Compare Gross, L. (1963). *Journal of Mathematics and Physics*, **5**, 687.

The object, now, is to show that the operators Z^α and $\overleftarrow{\partial}/\partial\bar{Z}_\alpha$ are equivalent to each other in their effect on the Hilbert space. That is to say, for any two functions $f(Z^\alpha)$, $g(Z^\alpha)$ of the type we have been considering, but with f and g homogeneous of respective degrees $-2s-3$ and $-2s-2$, we have (A_α being constant),

$$A_\alpha\langle g|Z^\alpha|f\rangle = \langle g|A_\alpha Z^\alpha f\rangle = \langle \bar{A}^\alpha \frac{\partial g}{\partial Z^\alpha}\Big|f\rangle = A_\alpha\,\langle g\Big|\frac{\overleftarrow{\partial}}{\partial \bar{Z}_\alpha}\Big|f\rangle \qquad (5.10)$$

We must verify the middle equality in (5.10). Thus, we have to show that, for all s,

$$\oint A_\lambda (W_\beta Z^\beta)^{2s-2} f(Z^\alpha)\left\{Z^\lambda(2s-1)\,\bar{g}(W_\alpha) + W_\rho Z^\rho \frac{\partial \bar{g}(W_\alpha)}{\partial W_\lambda}\right\}\mathscr{W}\wedge\mathscr{Z} = 0 \qquad (5.11)$$

[since $(2s-1)\,\Gamma(1-2s) = -\Gamma(2-2s)$]. (If $s = \frac{1}{2}, 1, 1\frac{1}{2}, \ldots$, then (5.11) 'automatically' vanishes and does not, in itself, imply (5.10) for these s values. But it will follow that (5.10) holds if we can verify (5.11) for *all* real s, since then a limiting argument on s will apply.) Now‡ (5.11) will hold if we can show that the expression under the integral sign is of the form $d\mathscr{X}$ for some 5-form $\mathscr{X}$. Set

$$\mathscr{X} = -\tfrac{1}{2}A_\lambda f(Z^\gamma)\,\bar{g}(W_\alpha)\,(W_\beta Z^\beta)^{2s-1}\,\epsilon^{\lambda\mu\nu\sigma}\,W_\mu\, dW_\nu \wedge dW_\sigma \wedge \mathscr{Z}$$

We wish to calculate $d\mathscr{X}$. By (5.4), we need only consider the terms involving derivatives with respect to W's. The calculation is straightforward if the identity

$$\begin{aligned}\epsilon^{\lambda\mu\nu\sigma}\, dW_\rho \wedge dW_\nu \wedge dW_\sigma &= \tfrac{1}{6}\epsilon^{\lambda\mu\nu\sigma}\,\epsilon_{\tau\rho\nu\sigma}\,\epsilon^{\tau\psi\chi\phi}\, dW_\psi \wedge dW_\chi \wedge dW_\phi \qquad (5.13)\\ &= \tfrac{1}{3}(\delta_\rho{}^\mu\,\epsilon^{\lambda\psi\chi\phi} - \delta_\rho{}^\lambda\,\epsilon^{\mu\psi\chi\phi})\times\\ &\quad \times dW_\psi\wedge dW_\chi \wedge dW_\phi\end{aligned}$$

is used, the terms in $A_\lambda\epsilon^{\lambda\mu\nu\sigma} dW_\mu \wedge dW_\nu \wedge dW_\sigma$ cancelling out. The result is that $d\mathscr{X}$ turns out to be precisely the term under the integral sign in (5.11), as required.

On the basis of (5.10) we can now write

$$Z^\alpha = \frac{\overleftarrow{\partial}}{\partial \bar{Z}_\alpha} \qquad (5.14)$$

where these are to be read as operators in the Hilbert space. The conjugate of (5.14) gives us

$$\bar{Z}_\alpha = \frac{\overrightarrow{\partial}}{\partial Z^\alpha} \qquad (5.15)$$

‡ See footnote on p. 92

(Planck's constant has been implicitly absorbed into the Z^α by the definition (5.1), since k has been chosen to be independent of s. This, then, implies a particular scaling for the $\phi_{A\ldots L}$ fields if the formulae of Section 3 are strictly adhered to.) We now have the commutation relation

$$\bar{Z}_\alpha Z^\beta - Z^\beta \bar{Z}_\alpha = \delta_\alpha{}^\beta \tag{5.16}$$

which may be compared with (4.37) (we, of course, also have $Z^\alpha Z^\beta = Z^\beta Z^\alpha$, $\bar{Z}^\alpha \bar{Z}^\beta = \bar{Z}^\beta \bar{Z}^\alpha$). This suggests a classical-quantum correspondence (Dirac, 1958)

$$i[\psi, \chi] = \psi\chi - \chi\psi \tag{5.17}$$

when ψ and χ are to be suitable quantum operators.

Note that the 'homogeneity operator' $Z^\alpha \partial/\partial Z^\alpha$ is just $Z^\alpha \bar{Z}_\alpha$. Since the eigenvalues of this operator are the values $-2s-2$, by (5.16) we can write

$$-\tfrac{1}{4}(Z^\alpha \bar{Z}_\alpha + \bar{Z}_\alpha Z^\alpha) = s \tag{5.18}$$

for the spin operator. The trace-free 'Hermitian' operator

$$\begin{aligned} E_\beta{}^\alpha &= Z^\alpha \bar{Z}_\beta - \tfrac{1}{4}\delta_\beta{}^\alpha Z^\gamma \bar{Z}_\gamma \\ &= \bar{Z}_\beta Z^\alpha - \tfrac{1}{4}\delta_\beta{}^\alpha \bar{Z}_\gamma Z^\gamma \end{aligned} \tag{5.19}$$

generates the *infinitesimal conformal transformations* of M (since if $p_\alpha{}^\beta$ is 'Hermitian' and trace-free: $\bar{p}_\beta{}^\alpha = p_\beta{}^\alpha$, $p_\alpha{}^\alpha = 0$; then the infinitesimal twistor transformation $Z^\alpha \to Z^\beta(\delta_\beta{}^\alpha + i\epsilon p_\beta{}^\alpha)$ is defined by the operator $i\epsilon p_\beta{}^\alpha E_\alpha{}^\beta = i\epsilon Z^\beta p_\beta{}^\alpha \partial/\partial Z^\alpha$, neglecting ϵ^2). Thus, the fifteen components of $E_\beta{}^\alpha$ include the *energy*, the three components of *momentum* and the six components of relativistic *angular momentum*, in addition to the five extra conservation laws which arise from the conformal invariance. If we wish to single out only the ten components of energy-momentum and angular momentum, then we employ the operator

$$F^{\alpha\beta} = E_\gamma^{(\alpha} I^{\beta)\gamma} \tag{5.20}$$

where $I^{\alpha\beta}$ is the (fixed) metric twistor (Penrose, 1967a) (whose only non-zero components, in the coördinate system of Section 2, are $I^{23} = -I^{32} = 1$; also, for $I_{\alpha\beta}$, just $I_{01} = -I_{10} = 1$ are non-zero).

It is of interest to see what the M-picture interpretation of the operators Z^α and $\partial/\partial Z^\alpha$ amounts to. Let $\phi_{AB\ldots L}$ be the spin s field corresponding to $f(Z^\alpha)$ as in Section 3. Then the spin $s-\frac{1}{2}$ field corresponding to $T_\beta Z^\beta f(Z^\alpha)$, where T_β is some constant twistor, is just

$$\psi_{B\ldots L} = \phi_{AB\ldots L}(-i\kappa^A + x^{AB'}\rho_{B'}) = -i\phi_{AB\ldots L}\tau^A \tag{5.21}$$

where, as in (2.14), the constant spinors κ^A, $\rho_{B'}$ are defined by $T_0 = -\kappa^1$, $T_1 = \kappa^0$, $T_2 = \rho_{1'}$, $T_3 = -\rho_{0'}$. Similarly, the spin $s+\frac{1}{2}$ field corresponding to $\bar{T}^\beta \partial f/\partial Z^\beta$ is

$$\chi_{AB\ldots LM} = i(2s+1)\bar{\rho}_{(M}\phi_{AB\ldots L)} - (\bar{\kappa}^{Q'} - ix^{PQ'}\bar{\rho}_P)\nabla_{Q'M}\phi_{AB\ldots L} \quad (5.22)$$
$$= -(s+\tfrac{1}{2})\{\nabla_{Q'(M}\tau^{Q'}\}\phi_{AB\ldots L)} - \bar{\tau}^{Q'}\nabla_{Q'M}\phi_{AB\ldots L}$$

One may verify directly that the $\psi_{B\ldots L}$ and $\chi_{A\ldots M}$ of (5.21), (5.22) satisfy the zero rest-mass equation (3.3), by virtue of (2.15): $\nabla_{P'(A}\tau_{B)} = 0$. In (5.22), $\phi_{A\ldots L}$ acts as a kind of potential field for $\chi_{A\ldots M}$. Particularly in the case $\bar{\rho}_P = 0$ (so τ^A is constant), this type of potential has been studied earlier (Penrose, 1965).

Finally, we may try to interpret, in the present formalism, the (conserved) 2-surface integrals of electromagnetic field ($s=1$) or of linearised gravitational field ($s=2$) which respectively define the total charge or total *mass*, *momentum*, and *angular momentum* of a *source* for the field. For this purpose we must allow ourselves to consider singularity regions for the function f which are more extensive than arose for the free-wave fields we had been considering previously. For the case of charge we have

$$Q = k_1 \oint f(Z^\alpha)\,\mathscr{Z} \quad (5.23)$$

where f is homogeneous of degree -4; and for mass, momentum and angular momentun,

$$G^{\alpha\beta} = k_2 \oint Z^\alpha Z^\beta f(Z^\alpha)\,\mathscr{Z} \quad (5.24)$$

where f is homogeneous of degree -6. Here k_1 and k_2 are constants, and the three-dimensional closed contour is chosen suitably so as to correspond to a region of free field surrounding the sources. Expressions (5.23) and (5.24) can both be obtained by directly translating the usual expressions in terms of the fields. (The arguments will be given elsewhere.) For free waves, the integrals (5.23), (5.24), of course, both must *vanish*. From this, we can see at once that the integral in (5.1) must necessarily vanish for such waves ($s=1,2$)—hence the necessity of the limiting procedure to define scalar product ($s=1, 1\frac{1}{2}, 2, \ldots$).

Note that (5.24) shows how the mass, momentum and angular momentum integrals can all be reduced to 'charge' integrals for certain spin-1 fields constructible from the given spin-2 field ϕ_{ABCD}. For, if we multiply (5.24) by a constant symmetric twistor $S_{\alpha\beta}$ then (5.24) reduces to (5.23), with $f(Z^\alpha)$ replaced by $S_{\alpha\beta}Z^\alpha Z^\beta f(Z^\alpha)$. In the

M-picture, this corresponds to calculating the 'charge' integral for the spin-1 field given by

$$\theta_{AB} = -\frac{k_2}{k_1}\phi_{ABCD}\,\sigma^{CD} \tag{5.25}$$

where σ^{CD} is the symmetric spinor, satisfying $\nabla^{P'(Q}\sigma^{CD)} = 0$, which corresponds to $S_{\alpha\beta}$ as in (2.17). For sources for which ϕ_{ABCD} can be derived from a potential, the 'magnetic' parts of (5.24) vanish. If $S_{\alpha\gamma}I^{\beta\gamma}$ is 'Hermitian' (i.e. $=\overline{S}^{\beta\gamma}I_{\alpha\gamma}$), then this means that there is zero 'magnetic charge' for θ_{AB} (as would follow if θ_{AB} is derivable from a potential). This condition, in terms of (5.24), is the 'Hermiticity' relation

$$G^{\alpha\gamma}I_{\beta\gamma} = \overline{G}_{\beta\gamma}I^{\alpha\gamma} \tag{5.26}$$

Equation (4.26) is the condition for $G^{\alpha\beta}$ $(= G^{\beta\alpha})$ to be of the form

$$G^{\alpha\beta} = B_{\gamma}^{(\alpha}I^{\beta)\gamma} \tag{5.27}$$

where $B_{\gamma}{}^{\alpha} = \overline{B}_{\gamma}{}^{\alpha}$, $B_{\alpha}{}^{\alpha} = 0$.

We may compare (5.20) with (5.27). The $F^{\alpha\beta}$ of (5.20) describes the total mass, etc., in the sense of *inertial* mass or *energy* content of the field. However, $G^{\alpha\beta}$ describes the total *active* mass, etc., as it appears as the *source* for the particular spin-2 field ϕ_{ABCD} [defined by the f of (5.24)]. This suggests a possible way that the actual *field equations* of general relativity might eventually be incorporated into the present formalism. For ϕ_{ABCD} to describe the gravitational field, we should expect something like

$$\langle F^{\alpha\beta}\rangle = G^{\alpha\beta} \tag{5.28}$$

A complication which will naturally arise, springs from the fact that ϕ_{ABCD} must, itself, contribute to $\langle F^{\alpha\beta}\rangle$. This would result in non-linearities of a type familiar in general relativity theory.

6. *Conclusions*

The twistor formalism appears to afford considerable scope for the expression of basic physical processes, several aspects of physics fitting unexpectedly naturally into the twistor framework. The development given here is an approximately 'historical' one. It was the desire to make the formalism fit in with general relativity which suggested the identification $\overline{Z}_{\alpha} = \partial/\partial Z^{\alpha}$ as a basis for an approach to quantization. This in turn led to the correct twistor expression for the zero rest-mass field Hilbert space scalar product (which had hitherto proved to be elusive). The identification emerged as being *consistent* with this scalar product. It is thus very tempting to believe that a link between space-time curvature and quantum processes may be supplied by the use of twistors. Then, roughly speaking, it is the continual slight 'shifting' of the interpretations of the quantum (twistor) operators which results in the curvature of space-time.

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